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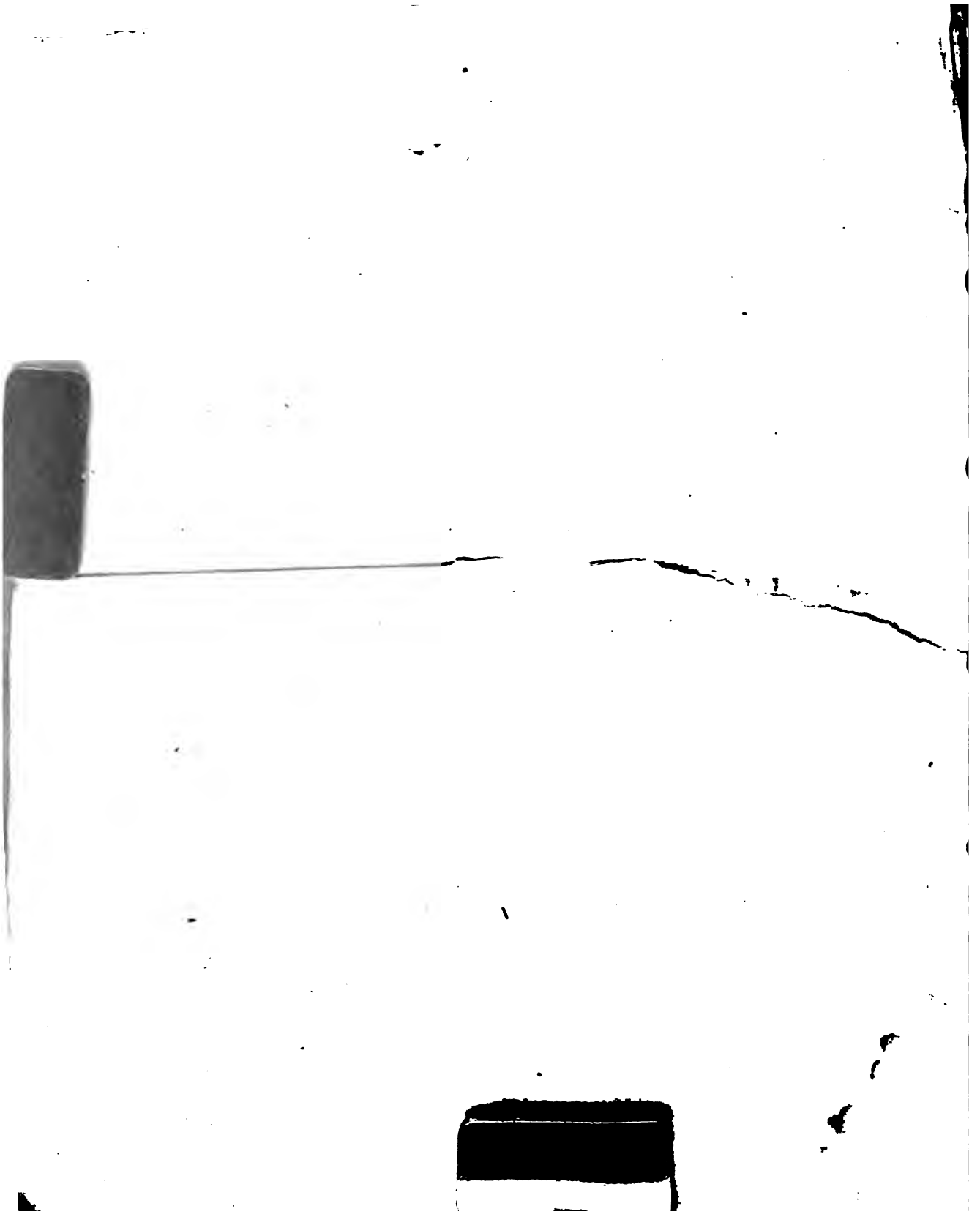

Euclides

THE

# Six First,

Together with the

# Eleventh and Twelfth Books

OF

# EUCLID's ELEMENTS,

Demonstrated after a New, Plain, and Easie METHOD.

By HENRY HILL, A. M.

——— *Labor omnia vicit*
*Improbus.* Virg.

*LONDON:*

Printed by WILLIAM PEARSON, and Sold by R. and J. BONWICKE, in St. *Paul's-Church-Yard*; F. FAYRAM, at the *South-Entrance* of the *Royal-Exchange*; and B. MOTTE, at the *Middle-Temple-Gate, Fleet-Street.* MDCCXXVI.

d

THE ELEMENTS of EUCLID being the moſt approv'd Book, for teaching the *Firſt Rudiments* of GEOMETRY; the Author hereof, the Reverend Mr. HENRY HILL, has taken Laudable Pains, to render it as plain and eaſie to the Reader, as the Nature of the Thing admits of.

*April* 8^vo^ 1724.

EDMUND HALLEY.
*Geom. Prof.*

Hist. of sci.
Bowes
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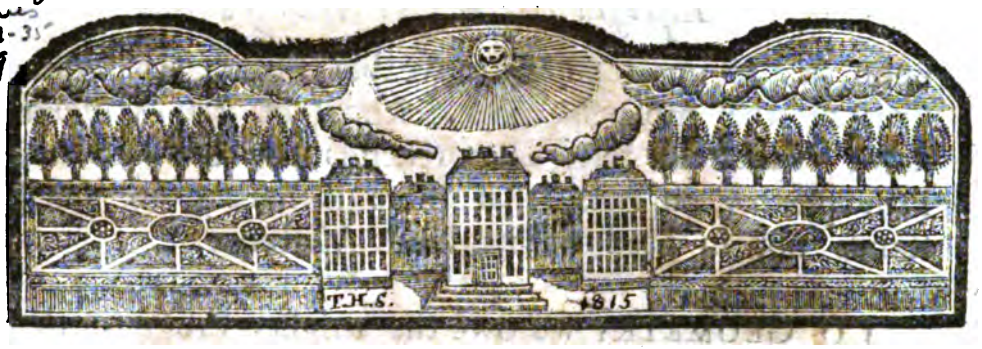


TO THE

Moſt High Puiſſant and Noble Prince

# CHARLES,

*Duke of* Somerſet, *Marquiſs and Earl of* Hertford, *Viſcount* Beauchampe de Hatche, *Baron* Seymour *of* Trowbridge, *Lord of the Honours of* Cockermouth, Petworth, &c. *Chancellour of the Univerſity of* Cambridge, *Knight of the moſt Noble Order of the* Garter, *and One of His Majeſty's moſt Honourable* Privy Council, &c. &c. &c.

9-10-35 Bowes

*May it pleaſe* YOUR GRACE.

AMONG all the manifold Sciences, which have hitherto been the Study and Delight of the Learned World, there is none that has ever been eſteemed of greater Extent, Uſe, and Profit, than that of GEOMETRY.

SO vaſtly Advantageous, Uſeful, and Beneficial, is the Knowledge thereof to Mankind in General, that upon

upon it, as upon an Immoveable Pillar, is Superſtructed the moſt Spacious Science of the *Mathematicks*; which ſo fully comprehends the Grounds and Principles of all Arts, that there is hardly One which ſtands not in need of its kind Aſſiſtance.

TO GEOMETRY we owe the mighty Progreſs, that has been already made in *Staticks*, *Hydroſtaticks*, *Opticks*, *Gnomonicks*, *Gunnery*, *Navigation*, and all other Inferiour Branches of the *Mathematicks*; nor are we leſs beholden to it for the Diſcovery of many New and Surpriſing *Phænomina*; for the Explanation of the Laws the Celeſtial Bodies obſerve in their Motions; and for the Inveſtigation of their true and Phyſical Cauſes.

THE ELEMENTS I here preſent YOUR GRACE, are the *Firſt Rudiments* of that Divine and Boundleſs Science, which, like the great LUMINARY, the SUN, extends it ſelf from Pole to Pole, and ſcatters its Light throughout all the Labyrinths of Humane Learning. They are, MY LORD, the *Firſt Principles* of that Sublime and Noble Science, which has had the good Fortune not only to be Univerſally receiv'd, but to be Admir'd alſo and Encourag'd, by the greateſt and moſt eminent Perſons, in all Ages and all Nations.

MAY it pleaſe YOUR GRACE, therefore to take the following Sheets into your Patronage and Protection; they being the *Firſt Rudiments* of that Science, which qualifies thoſe that underſtand it for all Employments,

ments, and renders them Ingenuous in all Profeſſions. May the Uſefulneſs of the Subject recommend them to YOUR GRACES Favour, whatever may be wanting, either in the Work or the Author.

I ſhould, MY LORD, offer at an Apology for preſuming to throw in my poor Endeavours, among your Great and Important Affairs, did not the Work it ſelf point out its Patron, and lay a more than ordinary Claim to YOUR GRACES Protection.

FOR, to whom, MY LORD, can I ſo properly Dedicate a Book calculated and intended for the Uſe and Benefit of the World, as to a Perſon who is always promoting the general Good of Mankind, and continually employ'd in the Service of the Publick? To whom can I ſo well inſcribe a Work of ſo much Intricacy, Labour, and Difficulty, as to a Perſon of YOUR GRACES Penetration, Candour, and Humanity? To whom can I with ſo much Juſtice ſend a Treatiſe of the *Firſt Principles* of almoſt every Art, as to One who is well acquainted with every Science, and adorn'd with Learning of every Sort?

YOU, MY LORD, are the moſt proper Perſon, to introduce the *Firſt Rudiments* of Arts and Sciences to the Publick, and to recommend them to the Learned World; Learning being the Employment and Entertainment of your noble Mind. Such is your Affection for Arts, that you countenance and encourage

every Artiſt; And ſo great is your Reputation for Learning, that you have the Honour to preſide over one of the moſt Learned Bodies in the Univerſe.

MUCH more, MY LORD, might be alledg'd, for laying this poor Performance at YOUR GRACES Feet, which is, I confeſs, infinitely unworthy of the Acceptance of ſo great a Perſon, and the Peruſal of ſo Judicious an Eye; but fearing leaſt I ſhould offend your Modeſty, and remembring how valuable every Moment muſt be to one of your High Birth, Rank, and Quality; I am the rather determin'd to wave the Proſecution, and to rely upon YOUR GRACES Candour and Condeſcenſion.

MAY the PRINCE of GEOMETRICIANS, the OMNIPOTENT CREATOR of the Univerſe, long preſerve YOUR GRACE, and may you long enjoy that Affluence, which, like the SUN it ſelf, affords Warmth and Life to all about you.

THIS, MY LORD, is the ſincere Wiſh, and hearty Prayer of him, who with the utmoſt Submiſſion and Reſpect, craves leave to Subſcribe himſelf,

YOUR GRACES,

*Moſt Obedient,*

*And moſt Humble Servant,*

Henry Hill.

# THE PREFACE.

*IT IS expected of those who appear in Publick, that they give the World a short Account, of the Usefulness of the Subjects upon which they Treat; of the chief Motives which induc'd them to Write; and of the Methods they have taken and made choice of in Writing.*

*THIS is that which is required of all Authors in General; but more especially of those, who undertake Subjects, which are not only Intricate and Difficult in themselves, but which have been often handled, and that by Men of the greatest Genius's, Judgments, and Abilities.*

*THE Contents then of the following Sheets, being the* First Rudiments *of* Geometry, *the Usefulness of the Subject, will, I presume, appear, from that of* Geometry *itself; there being no Science whatsoever, that can afford greater Delight or Entertainment to the Mind; that is so Extensive and Unbounded in its Use and Application; or that can lay a more just Claim, either to the Invention or the Improvement of all useful Mechanick Arts.*

*THE chief Motive which induc'd me to Write, was the Usefulness of the Subject upon which I Treat; it being the infinite Advantages which accrew from* Geometry *to Mankind in General, that led me to compose the ensuing Treatise, and to set out every Demonstration in such a regular, plain, and easie Manner, as might neither burthen the Memory, nor perplex the Mind; but that the Young Student might proceed with Ease and Cheerfulness, in the* First Rudiments *of that great and spacious Science, which is the Basis and Foundation of all Parts and Branches of*

*of the* Mathematicks, *and which intitules thoſe that underſtand it to ſuch Penetration and Exactneſs, as fits and prepares them for all Employments.*

*THE Method I have taken in Writing, is entirely new; wholly different from any that has hitherto been made choice of by any of the Interpreters of* EUCLID, *and ſuch as does not ſo much require Intent and ſevere Thinking, as a bare and eaſie Inſpection.*

*THE better to avoid Obſcurity and Confuſion, and that painful Attention, which we have a general Averſion to, and which diſturbs and diſtracts the Mind; I have ſet each, and every Demonſtration out in Lines, or Steps; ſo that the ſeveral Medium's that are neceſſary, in Order to inveſtigate the Truth of many Propoſitions, might no ways puzzle or perplex the Young* Geometrician; *but that he might have a clear and perfect Notion and Idea, of the moſt Intricate Demonſtrations in the* ELEMENTS *of* GEOMETRY, *and proceed with Courage and Delight, in queſt of the* Firſt Principles *of real and true* Philoſophy.

*BESIDES that of Steps, which runs throughout the Book, I have (in order to facilitate the Work) had recourſe ſometimes to one Method, ſometimes to another, according as I judg'd this or that of them, fitteſt to Demonſtrate the Thing in Hand; but having chiefly, and for the moſt Part, made uſe of* Algebra, *where I thought it convenient to deviate from the Original, I have, for that Reaſon, prefixt as much of* Literal Arithmetick, *as is any where required in the following Demonſtrations.*

*WHAT Opinion the World may have of theſe my Endeavours, after ſo many Ingenuous and Elaborate Performances on the ſame Subject, Time muſt teach me; all that I have to requeſt, is, that the following Sheets may be impartially perus'd, and not too rigorouſly cenſur'd, but that the Candid Reader would excuſe and paſs over any ſlight Error, that may poſſibly occur in the enſuing Treatiſe.*

*The*

*The Explanation of the Signs, Characters, and Abbreviations of Words, uſed in this BOOK.*

| | | |
|---|---|---|
| $+$ | Signifies | More, or to be added. |
| $-$ | | Leſs, or to be ſubtracted. |
| $\times$ | | Multiplyed by. |
| $\div$ | | Divided by. |
| $\frac{A}{B}$ | | A divided by B. |
| $1+2$ | | The firſt Step added to the ſecond Step. |
| $1+2AB$ | | The firſt Step added to the Quantity 2 A B. |
| $1-2$ | | The firſt Step wanting the ſecond Step. |
| $1-2AB$ | | The firſt Step wanting the Quantity 2 A B. |
| $1\times 2$ | | The firſt Step multiplyed by the ſecond Step. |
| $1\times\overline{2}$ | | The firſt Step multiplyed by the abſolute Number 2. |
| $1\div 2$ | | The firſt Step divided by the ſecond Step. |
| $1\div\overline{2}$ | | The firſt Step divided by the abſolute Number 2. |
| $=$ | | Equality, or equal to. |
| $\sqsubset$ | | Majority, or greater than. |
| $\sqsupset$ | | Minority, or leſs than. |
| $\parallel$ | | Parallel to. |
| $\because$ | | Therefore. |
| e. | | From. |
| $\angle$ | | Angle. |
| $\angle^s$ | | Angles. |
| r$\angle$ | | Right-angle. |
| r$\angle^s$ | | Right-angles. |
| $\triangle$ | | Triangle. |
| $\triangle^s$ | | Triangles. |
| $\perp$ | | Perpendicular, and Perpendicular to. |
| $\perp^s$ | | Perpendiculars. |

| | | |
|---|---|---|
| □ | Signifies | Parallelogram. |
| □' | | Parallelograms. |
| ⊕ | | Involution, or the Raiſing or Producing of Powers, from any propoſed Root. |
| ω | | Evolution, or the Extracting of Roots from any given Power. |
| ⊕ 2 | | The Squaring of any Quantity, or Involving of it to the ſecond Power. |
| ω 2 | | The Extracting of the Square Root of any Quantity. |
| A' | | The Cube of A. |
| A'B | | The Cube of A B. |
| q | | Square. |
| \|2 | | Square. |
| √ | | Square Root. |
| A:B::C:D. | | As A is to B, ſo is C to D. |
| Ax. | | Axiom. |
| Poſt. | | Poſtulate, or Petition. |
| Def. | | Definition. |
| Conſt. | | Conſtruction. |
| Sub. | | Subſtitution, or the placing of one Quantity in the Room of another, which is equal to it. |
| Hyp. | | Hypotheſis, or Suppoſition. |

# *A LIST of the Subſcribers to, and Encouragers of this WORK.*

## A

The Right Honourable the Counteſs Dowager of *Aylesford.*
William Andrews, *Eſq;* 2 Books.
Anonymus.

## B

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Edward Bray, *Eſq;* 3 Books.
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John Bridges, *Eſq;*
Francis Brickenden, *Eſq;*
The Rev. *Mr.* Billingſley, late Archdeacon of *Surry.*
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*Mrs.* Etterick.
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*Mr.* John Ellam.

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The Honourable Henry Finch, *Eſq;*
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THE

THE

# SIX FIRST,

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ELEVENTH and TWELFTH BOOKS

OF

# EUCLID's ELEMENTS,

Demonſtrated after a New, Plain, and Eaſie

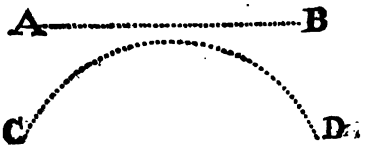
METHOD.

## BOOK I.

### DEFINITIONS.

1 A *Geometrical Point*, is that which has no Parts, or at leaſt is conſidered as ſuch, and which of Conſequence is indiviſible.

2 A *Line*, whether *Straight* or *Crooked*, is a Length without either Breadth or Thickneſs, and is conceiv'd to be generated by the Motion of a Point; as A B, C D.




Becauſe a *Line* hath Length, it is called a Quantity of one Dimenſion, and is ſuppoſed to conſiſt of an infinite Number of Points.

3 The *Ends*, or *Limits* of a Line, are Points.

4 A *Right-line*, is that whereof the Points are equally placed between its Two Extremities; as A B.

Whence it follows, that a *Curve-line*, is that which has not all its Points placed equally between its two Extremities; becauſe ſome are elevated above, and ſome ſubſide below others; as C D.

5 A *Superficies* or *Surface*, is that which hath Length and Breadth without Thickneſs, and is conceiv'd to be generated by the Motion of a Line, as a Line is by the Motion of a Point.

Thus, if A B, were equally mov'd along the Line A C, to C D; it would form the *Superficies* or *Surface* A B C D.

Becauſe a *Superficies* or *Surface* hath Length and Breadth, it is called a Quantity of two Dimenſions, and is ſuppoſed to conſiſt of an infinite Number of Lines.

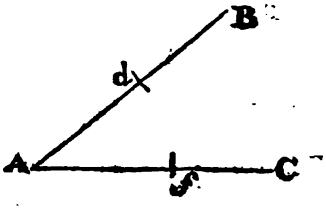
The

6 The *Extremities*, or *Ends* of a Superficies or Surface, are Lines.

7 A *Plane-Surface* or *Plane*, is that which hath all its Right-lines equally placed between its Extremities; ſo that one does not riſe higher, or ſubſide lower than the other; as A B C D.

8 A *Plane-Angle*, is the Inclination of two Lines the one to the other, the one touching the other in the ſame Plane, yet not lying in the ſame Straight Line; as B A C.

And according as ſuch Lines are inclin'd to each other, that is, ſtand nearer or further off each other, the *Angle* is ſaid to be leſſer or greater, whether the Lines which include it be ſhort or long.

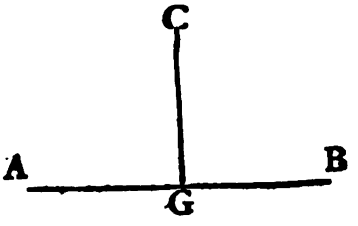



Thus, the Lines A *d*, A *f*, include the ſame *Angle*, as the Lines A B, A C; notwithſtanding that A B is longer than A *d*, and A C than A *f*.

9 All Angles, whether Right, Obtuſe, or Acute, which are included between Right-lines, are called *Right-lin'd Angles*.

10 When a Right-line CG, ſtandeth on a Right-line AB, and maketh the Angles CGA, CGB, on either Side thereof, equal one to the other, then both thoſe equal Angles are *Right-Angles* and the Right-line CG, which ſtandeth on the other AB, is term'd a *Perpendicular* to that other AB, whereon it ſtandeth.

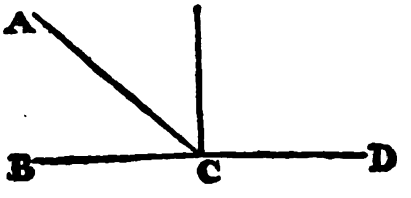
*Note.* When ſeveral Angles meet at the ſame Point, as at G, each particular Angle is deſcribed by three Letters; whereof the middle Letter ſheweth the angular Point, and the two other Letters, the Lines that make the Angle.




Thus, the Angle which the Right-lines CG, AG make at G, is called CGA, or AGC.

11 An *Obtuſe-Angle* is that which is greater than a Right-Angle; as ACD.

12 An *Acute-Angle* is that which is leſs than a Right-Angle; as ACB.

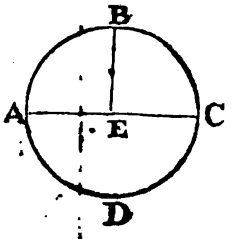



13 A *Limit*, or *Term*, is the End of any Thing.

14 A *Figure* is any Space or Quantity of two or three Dimenſions, comprehended under, or bounded every Way by one or more Terms.

15 A *Circle* is a plane Figure, terminated by a Boundary of one Line only, which is called its *Periphery* or *Circumference*, as A B C D A, within which is a Point, as E, called the *Center*; from which all the Right-lines, *Semidiameters*, or *Radij*, E A, E B, E C, *&c.* drawn to the *Circumference*, are equal to one another.

16 The *Center* therefore *of a Circle* is a Point within its Circumference, from which all Right-lines drawn to that Circumference, are equal among themselves; as E.

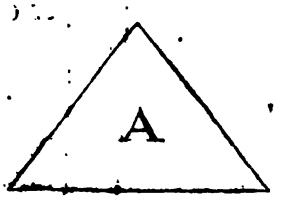



17 The *Diameter of a Circle*, is any Right-line drawn through the Center, and terminated by the Circumference on each Side, dividing the Circle into two equal Parts; as A C.

18 A *Semicircle*, is a Figure which is contained under the Diameter, and under that Part of the Circumference which is cut off by the Diameter; as A B C.

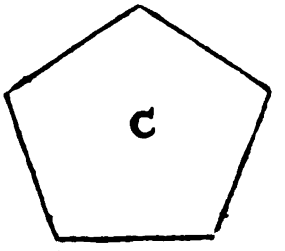
19 *Right-lin'd Figures*, are such as are contained under Right-lines.

20 *Three-sided Figures*, or *Triangles*, are such as are contain'd under three Right-lines; as A.



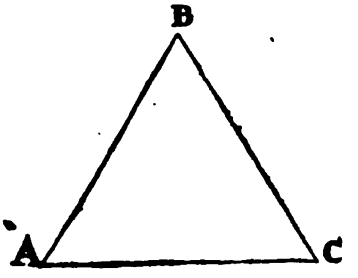
21 *Four-ſided*, or *Quadrilateral Figures*, are ſuch as are contained under Four Right-lines; as B.

22 *Many-ſided Figures*, or *Polygons*, are ſuch as are contain'd under more Right-lines than four; as C.



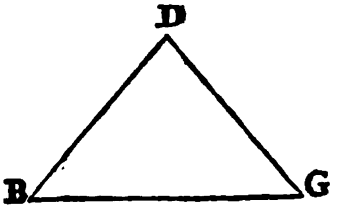

23 An *Equilateral Triangle*, is that which has all its Three Sides equal; as A B C.

That is, A B = B C = A C.

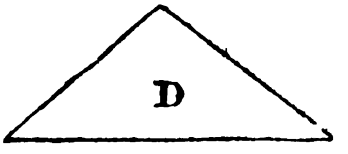



24 An *Iſoſceles Triangle*, is that which hath only Two of its Sides equal; as D B G.

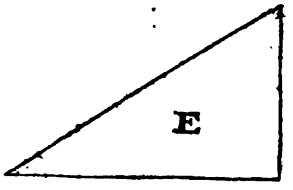
That is, D B = D G; but the Third B G, may be either greater or leſs, as Occaſion requires.




25 A *Scalenous Triangle*, is that which hath all its Three Sides unequal; as D.



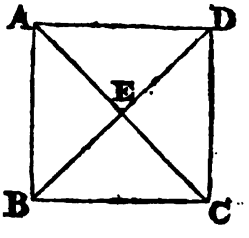
26 A *Right-angled Triangle*, is that which hath one Right-angle; as E.




27 An *Obtuse-angled Triangle*, is that which hath One of its Angles Obtuse; as D.

28 An *Acute-angled Triangle*, is that which hath all its Angles Acute; as ABC.

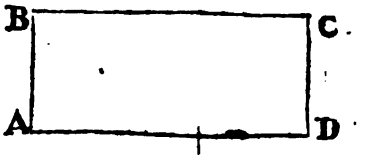
29 A *Square*, is a Quadrilateral Figure contain'd under Four equal Sides, all perpendicular to one another; as ABCD.



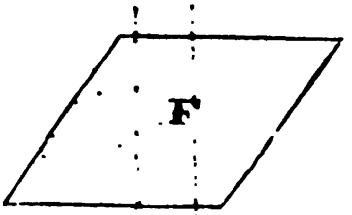

That is, when AB = BC = CD = AD, and the Angles A, B, C, D, are equal, then it is usually called a *Geometrical Square*.

A Right-line drawn from any Angle of a *Square*, to the opposite One, as AC, DB, is called its *Diagonal*; and the Point where Two such *Diagonals* intersect each other, is called its *Center*; as E. 22/0

30 A *Rectangle*, or *Right-angled Parallelogram*, (often called an *Oblong*, or *Long Square*,) is a Figure that hath Four Right-Angles, and its Two opposite Sides equal, *viz.* BC = AD, and BA = CD.



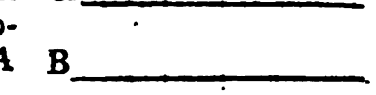
31 A *Rhombus*, or *Diamond-like Figure*, is that which hath Four equal Sides, but no Right-Angle. That is, a *Rhombus* is a Square moved out of its right Poſition; as F.




32 A *Rhomboid*, or *Diamond-like Figure*, is that whoſe oppoſite Sides, and oppoſite Angles are equal; but has neither equal nor Right-Angles; as G.

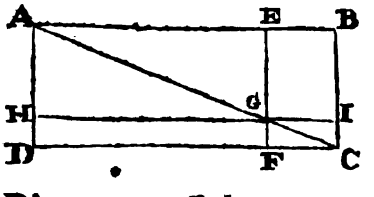
33 All Four-ſided Figures, which differ from thoſe before mentioned, are called *Trapezia*, or *Tables*; as H.

34 *Parallel*, or *Equidiſtant Right-lines*, are ſuch as being in the ſame Superficies, if infinitely produced, would never meet; as A and B.




35 A *Parallelogram*, is a Quadrilateral Figure, whoſe oppoſite Sides are Parallel, or Equidiſtant; as G.

36 In a Parallelogram A B C D, when a Diameter A C, and two Lines E F, H I, parallel to the Sides, cutting the Diameter in one and the ſame Point G, are drawn, ſo that the Parallelogram be divided by them into Four Parallelograms; thoſe Two, D G, G B, thro' which the Diameter paſſeth not, are called *Complements*; and the other Two H E, F I, through which the Diameter paſſeth, *the Parallelograms ſtanding about the Diameter.*




A

37 A *Problem*, is, when ſomething is propoſed to be done or effected.

38 A *Theorem* is, when ſomething is to be demonſtrated.

39 By *Demonſtration* is underſtood the higheſt Degree of Proof that human Reaſon is capable of attaining to, by a Train of Arguments, deduced or drawn from ſuch plain Axioms, and other ſelf-evident Truths, as cannot be denied by any One that conſiders them.

40 A *Corollary*, is a *Conſectary*, or ſome conſequent Truth gained from a preceding Demonſtration.

41 A *Lemma*, is the Demonſtration of ſome Premiſs laid down, or propoſed as a Preparative to obviate and ſhorten the Proof of the Theorem under Conſideration.

42 A *Scholium*, is a brief Commentary, or Obſervation made upon ſome precedent Diſcourſe.

## POSTULATES or PETITIONS.

1 That a *Right-line* may be drawn from any one given *Point* to another.

2 That a *Right-line* may be *Produced*, *Increaſed*, or made *Longer*, from either of its *Ends*.

3 That upon any given *Point* (or *Centre*) and with any given *Diſtance*, (*viz.* with any *Radius*) a *Circle* may be deſcribed.

## AXIOMS.

1 Quantities equal to the ſame Third, are alſo equal one to the other.
As $A = B = C$. Therefore $A = C$. Or therefore A, B, C, are equal the one to the other.

2 Any Quantity that is leſs or greater than either of two equal ones, is alſo leſs or greater than the other.
As $C \sqsupset$ or $\sqsubset A$, and $A = B$. Therefore $C \sqsupset$ or $\sqsubset B$.

3 If of two equal Quantities the one is greater or leſs than a Third, the other ſhall be alſo greater or leſs than that Third.
As $A = B$, and $A \sqsubset$ or $\sqsupset C$. Therefore $B \sqsubset$ or $\sqsupset C$.

4 A Quantity is equal to another Quantity, when it is neither greater nor leſs than that Quantity.
As $A = B$, when it is neither $\sqsubset$ nor $\sqsupset B$.

5 A Quantity is greater than another Quantity, when it is neither equal nor leſs.
As $A \sqsubset B$, when it is neither $=$ nor $\sqsupset B$.

6 A Quantity is leſs than another Quantity, when it is neither equal nor greater.
As $A \sqsupset B$, when it is neither $=$ nor $\sqsubset B$.

7 Quantities that are double each of the ſame Quantity, or of equal Quantities, are equal among themſelves.
As $A = 2C$, and $B = 2C$. Therefore $A = B$.
Underſtand the ſame of Triple, Quadruple, &c.

Quanti-

8 Quantities that are each one half of the ſame Quantity, or of equal Quantities, are equal among themſelves.
As $A = \frac{1}{2}C$, and $B = \frac{1}{2}C$. Therefore $A = B$.
Underſtand the ſame of Subtriple, Subquadruple, &c.

9 If the Roots of any two Quantities are equal, their Squares are alſo equal.
As $A = B$. Therefore $AA = BB$.

10 If the Squares of any two Quantities are equal, their Square Roots are alſo equal.
As $AA = BB$. Therefore $A = B$.

11 If to equal Quantities you add equal Quantities, their Sums will be equal.
Thus $6 = 6$. And $6 + 3 = 6 + 3 = 9$.

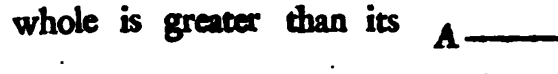
12 If from equal Quantities, you ſubtract equal Quantities, their Remainders will be equal.
Thus, $6 = 6$. And $6 - 3 = 6 - 3 = 3$.

13 If equal Quantities, be multiplied by equal Quantites, their Products will be equal.
Thus, $6 = 6$ and $6 \times 3 = 6 \times 3 = 18$.

14 If equal Quantities, be divided by equal Quantities, their Quotients will be equal.
Thus, $6 = 6$. And $6 \div 3 = 6 \div 3 = 2$.

15 If to unequal Quantities, you add equal Quantities, their Sums will be unequal.
Thus, $8 > 4$. And $8 + 2 = 10$, is $> 4 + 2 = 6$.

16 If from unequal Quantities, you ſubtract equal Quantities, their Remainders will be unequal.
Thus, $8 \sqsubset 4$. And $8 - 2 = 6$, is $\sqsubset 4 - 2 = 2$.

17 Every whole is greater than its Part.


Thus, the whole Line AB, is greater than its Part AC.
The ſame is to be underſtood of Superficies and Solids.

18 Every whole is equal to all its Parts taken together.
Thus, the whole Line AB, is equal to all its Parts AC + CB.
The ſame is alſo true in Superficies and Solids.

19 Quantities which every Way agree, are equal.

That is, if two Lines being placed one upon the other, do ſo agree, as that all the Parts of the one correſpond exactly to all the Parts of the other; ſo that neither ſurpaſſes (or is greater or leſs than) the other, thoſe two Lines are equal.
The ſame may be ſaid of two Angles, of two Surfaces, or of two Solids, when one being placed upon the other, or ſuppoſed to penetrate the other, neither of them ſurpaſſes the other.

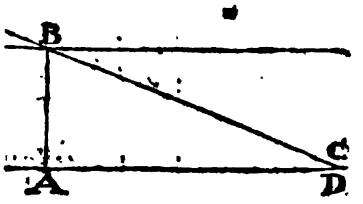
20 Two Right-lines cannot comprehend, (or include) ſpace, or conſtitute a Figure.
For it is evident, that two Right-lines meeting one another, can only make an Angle, which is not a Figure.

21 All Right-Angles are equal to one another.

This follows from the Definition of a Perpendicular, which ſuppoſes, that it makes on the Line on which it falls two equal Angles, which we call Right-Angles.

If

22 If a Right-line B A, falling on two Right-lines A D, C B, make the internal Angles
9 on the ſame Side, B A D, A B C, leſs than two Right-angles, thoſe two Right-lines produced ſhall meet on that Side, where the Angles are leſs than two Right-angles.




## *Of Tracing the Steps us'd in bringing Quantities to an Æquation.*

The Method of Tracing the Steps, us'd in bringing Quantities to an *Æquation*, is beſt perform'd by Regiſtring the ſeveral Operations with Figures and Signs placed in the Margin of the Work, according as the ſeveral Operations require.

For Inſtance, If it were requir'd to ſet down, and regiſter the Sum of the two Quantities A, and B, the Work will ſtand,

Thus 1 A
2 B
1 + 2 3 A + B

First ſet down the propos'd Quantities, A and B, over-againſt the Figures 1, 2, in the ſmall Column, which are here called Steps, and againſt 3, (the third Step) ſet down their Sum, *viz.* A + B. Then againſt that third Step ſet down 1 + 2 in the Margin; which denotes that the Quantities againſt the firſt and ſecond Steps are added together, and that thoſe in the third Step are their Sum.

To illuſtrate this in Numbers, ſuppoſe A = 6, and B = 3.

Then it will be,

Thus
1 A = 6
2 B = 3
1 + 2 3 A + B = 6 + 3 = 9. the Sum of 6 and 3.

Again, if it were requir'd to ſet down their Difference.

Then it will be,

Thus
1 A = 6
2 B = 3
1 − 2 3 A − B = 6 − 3 = 3. the Difference between 6 and 3.

Moreover, if it were requir'd to ſet down their Product.

Then it will be,

Thus
1 A = 6
2 B = 3
1 × 2 3 A × B = 6 × 3 = 18. the Product of 6 into 3.

*Laſtly*, if it were requir'd to ſet down their Quotient.

Then it will be.

Thus
1 A = 6
2 B = 3
1 ÷ 2 3 $\frac{A}{B} = \frac{6}{3} = 2$. the Quotient of 6 by 3.

## *Addition of* Algebraick Integers.

### CASE I.

If the Quantities are alike, and have like Signs; add the Co-efficients or prefix'd Numbers together; and to their Sum adjoyn the Quantities with the ſame Sign.

EXAM-

## EXAMPLES.

| | | | | | | |
|---|---|---|---|---|---|---|
| Thus 1 | + 1 A | — 1 A | 5 B | — 7BC | 3A + 12 | YY + 2X — 2 |
| 2 | + 1 A | — 1 A | 8 B | — 10BC | 2A + 6 | YY + 3X — 5 |
| 1 + 2 3 | + 2 A | — 2 A | 13 B | — 17BC | 5A + 18 | 2YY + 5X — 7 |

The Reaſon of theſe Additions will appear, if we conſider that if to 1 A, which may repreſent one Crown, be added 1 A, or one Crown, the Sum will be 2 A, or two Crowns; alſo if to — 1 A, which may repreſent the Want or Debt of one Crown, be added — 1 A, or the Want or Debt of another Crown, the Sum will be — 2 A, or the Want or Debt of two Crowns.

## CASE II.

If the Quantities are alike, and have unlike ſigns; ſubtract the Co-efficients from each other, and to their Difference joyn the Quantities with the Sign of the greater Quantity.

## EXAMPLES.

| | | | | | |
|---|---|---|---|---|---|
| Thus 1 | A | — 5 A | 3 BC | 7 A — 5 B | — 7 ABD + 15 |
| 2 | — A | + 3 A | — 2 BC | — 3 A + 7 B | + 9 ABD — 12 |
| 1 + 2 3 | 0 | — 2 A | BC | 4 A + 2 B | 2 ABD + 3 |

The Reaſon of the Operations in this *Caſe* will be eaſily underſtood, if we conſider that to add a Negative, is to take away a Poſitive; therefore to connect a Negative and a Poſitive, is to make the one to deſtroy the other.

Thus,

Thus, if A has 600*l.* and owes 400*l.* 'tis plain that the Sum, or his worth is but 200*l.*

And if A has 600*l.* and owes 900*l.* then the Worth is — 300*l.* or 300*l* worse than Nothing.

## CASE III.

When the Quantities are unlike, ſet them all down without altering their Signs.

### *EXAMPLES.*

| | | | | | |
|---|---|---|---|---|---|
| Thus | 1 | 4 M | A | 6C+7DC | 5X+Y+Z |
| | 2 | 5 H | —B | 4A | B +C—M |
| 1 + 2 | 3 | 4M+5H | A—B | 6C+7DC+4A | 5X+Y+Z+B+C—M |

The Reaſon of the Operations in this *Caſe* will appear, if we ſuppoſe M to repreſent Miles and H Hours; for then 4 M + 5 H, that is, 4 Miles added to 5 Hours, will make neither 9 Miles nor 9 Hours.

### *Subduction of* Algebraick Integers.

### General Rule.

Change all the Signs of the Subtrahend; (*viz.* of thoſe Quantities which are to be ſubtracted) or ſuppoſe them in your Mind to be chang'd. Then add all the Quantities together, as before in Addition, and their Sum will be the true Remainder or Difference requir'd.

*EXAM-*

## EXAMPLES.

| | | | | | |
|---|---|---|---|---|---|
| Thus | 1 | $+A$ | $-B$ | $Y$ | $-X$ | $3AA-5A+7$ |
| | 2 | $+A$ | $-B$ | $-Y$ | $X$ | $AA+A-2$ |
| 1 — 2 | 3 | $A-A$ | $-B+B$ | $Y+Y$ | $-X-X$ | $2AA-6A+9$ |
| | | 0 | 9 | 2 Y | $-2X$ | |

The Reaſon of this *General Rule* will appear, if we conſider.

1ſt. That if a Poſitive be taken from its Poſitive, that is, $+A$ from $+A$, the Remainder will be 0; ſince to take away the Affirmation of any Quantity A, is to make it Negative, or $-A$; ſo that the one will deſtroy the other.

2dly. That if a Negative be taken from its Negative, that is, $-B$ from $-B$, the Remainder will be 0; ſince to take away the Negation of any Quantity B, is to make it Affirmative, or $+B$; ſo that the one will deſtroy the other.

3dly. That if a Negative be taken from its Poſitive, that is, $-Y$ from $+Y$, the Remainder will be double of that Poſitive, *viz.* 2 Y; ſince to ſubtract or take away a Privation or Negative, is to add that very Thing the Privation of which you take away.

4thly. That if a Poſitive be taken from its Privative, that is, $+X$ from $-X$, the Remainder will be double the Privative One, *viz.* $-2X$; ſince by taking away a Poſitive One, there neceſſarily ariſes a new Privation, which will double that you had before.

## *Multiplication of* Algebraick Integers.

### CASE I.

When the Quantities have like Signs, and no Co-efficients, ſet or join them together; and prefix the Sign + before them, and that will be their Product.

### *EXAMPLES.*

| | | | | | | |
|---|---|---|---|---|---|---|
| Thus | 1 | A | − A | A + B | − A − B | A + R |
| | 2 | B | − B | D | − D | R |
| 1 × 2 | 3 | AB | + AB | AD + DB | + AD + DB | AR + RR |

### CASE II.

If there be Co-efficients; multiply them, and to their Product adjoin the Quantities ſet together as before.

### *EXAMPLES.*

| | | | | | | |
|---|---|---|---|---|---|---|
| Thus | 1 | 5 A | − 6 D | 3 A + 2 B | 2 A + 5 R | A + B |
| | 2 | 3 B | − 7 B | 6 | 3 R | 8 B |
| 1 × 2 | 3 | 15 AB | + 42 BD | 18 A + 12 B | 6 AR + 15 RR | 8 AB + 8 BB |

The Reaſon of theſe Operations will appear, if we conſider that to multiply an Affirmative by an Affirmative Quantity, is ſo often to repeat the Affirmation of it; and conſequently + into + will give + in the Product.

But

But to multiply a Negative by a Negative Quantity, is ſo many Times to deny that Negation.

And ſo many Times to deny the Negation of a Thing, is ſo many Times to affirm that Thing.

Therefore — into — will give + in the Product.

## CASE III.

When Quantities have unlike Signs, join them and the Product of their Co-efficients together, (as before) but prefix the Sign — before them.

## *EXAMPLES.*

| | | | | | | |
|---|---|---|---|---|---|---|
| Thus | 1 | A | — 6D | — 4A — 7B | —A—B | A+B |
| | 2 | —B | + 7B | 3F | C | —C |
| 1 × 2 | 3 | — AB | —42BD | —12AF—21BF | —AC—BC | —AC—BC |

The Reaſon of theſe Multiplications may be eaſily underſtood, if we conſider that to multiply an Affirmative by a Negative, is ſo often to repeat the Negation of it; and conſequently — into +, or + into —, will give — in the Product.

*Note:* If the Multiplier conſiſts of ſeveral Terms, then every one of thoſe Terms muſt be multiplied into all the Terms of the Multiplicand: And the Sum of thoſe particular Products, will be the Product requir'd.

## EXAMPLES.

| | | | |
|---|---|---|---|
| Thus | 1 | A + B − D | 7B + 5D |
| | 2 | A − B | 3A − 5F |
| 1 × A | 3 | AA + AB − AD | 21AB + 15AD |
| 2 × −B | 4 | − AB − BB + BD | − 35BF − 25DF |
| 3 + 4 | 5 | AA − AD − BB + BD | 21AB + 15AD − 35BF − 25DF |

### *Diviſion of* Algebraick Integers.

### CASE I.

When the Quantities in the Dividend, have like Signs to thoſe in the Diviſor, and no Co-efficients in either; caſt off or expunge all the Quantities in the Dividend, that are like thoſe in the Diviſor; and ſet down the other Quantities with the Sign + for the Quantities requir'd.

## EXAMPLES.

| | | | | | | |
|---|---|---|---|---|---|---|
| Thus | 1 | AB | − AB | AD + BD | − AD − BD | CXXXZYYY |
| | 2 | B | − B | D | − D | Y |
| 1 ÷ 2 | 3 | A | + A | A + B | + A + B | CXXXZYY |

### CASE II.

When the Quantities in the Dividend, have unlike Signs to thoſe in the Diviſor; then ſet down the Quotient Quantities found as before, with the Sign − before them.

EXAM-

## EXAMPLES.

| | | | | | |
|---|---|---|---|---|---|
| Thus | 1 | +AB | −AB −BD | ABC+BCD | XX+ZX |
| | 2 | − B | +B | −BC | −X |
| 1 ÷ 2 | 3 | − A | −A −D | −A −D | −X−Z |

## CASE III.

If the Quantities in the Dividend and Diviſor, have Co-efficients; divide the Numbers (as in common *Arithmetick*) and to their Quotient adjoin the Quotient Quantities.

## EXAMPLES.

| | | | | | | |
|---|---|---|---|---|---|---|
| Thus | 1 | 15AB | 9BC | 42DB | 12AF−21BF | AB |
| | 2 | 3B | −9BC | −7B | 3F | AB |
| 1 ÷ 2 | 3 | 5A | −1 | −6D | 4A − 7B | +1 |

## CASE IV.

When the Quantities in the Diviſor cannot be exactly found in the Dividend; then ſet them both down like a Vulgar Fraction.

## EXAMPLES.

| | | | | | |
|---|---|---|---|---|---|
| Thus | 1 | A | 6BC | 5B+AA | 8ADC | XX+YY+Z |
| | 2 | B | 3D | 5D+5C | 4ABC | 3B+2D |
| | | A | 6BC | 5B+AA | 8ADC | XX+YY+Z |
| 1 ÷ 2 | 3 | — | — | — | — | — |
| | | B | 3D | 5D+5C | 4ABC | 3B+2D |

Diviſion being the Converſe of Multiplication, the Truth of theſe Operations will appear, by multiplying each Diviſor into its reſpective Quotient, which (according to the Rules laid down in Multiplication) will produce the Dividend. As in common *Arithmetick*.

### *Involution of* Quantities.

A Quantity multiplied into it ſelf any Number of Times, is ſaid to be Involv'd, the Products ariſing are called Powers, and the Quantity multiplied, a Root.

Thus $\begin{cases} A \times A = AA \\ A \times A \times A = AAA \text{ or } A^3. \end{cases}$

Here A is the Root or firſt Power; $A^2$, $A^3$, *&c.* The Second, Third, *&c.* Powers of it.

Involution of Compound Quantities is perform'd in the ſame Manner, due regard being had to their Signs and Co-efficients, if there be any.

For

For Inſtance, Suppoſe A + B were given to be Involv'd to the ſecond Power.

| | | |
|---|---|---|
| | 1 | A + B<br>A + B |
| 1 × A | 2 | AA + AB |
| 1 × B | 3 | + AB + BB |
| 2 + 3 | 4 | AA + 2AB + BB the Square of A + B. |

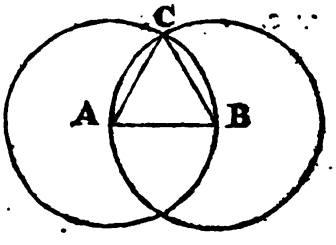
*Note.* I would adviſe the Learner, to make himſelf thoroughly acquainted with the preceding *Definitions*, *Poſtulates*, *Axioms* and *Algebraick Arithmetick*; as alſo with the *Definitions* prefixt to the ſubſequent Books.

## PROPOSITION I.

*Upon a finite Right-line* AB, *to deſcribe an Equilateral Triangle* ABC.

### PREPARATION.

From the Centers A and B, at the Diſtance of AB, or BA, deſcribe (*poſt.* 3.) two Circles cutting each other in the Point C; from whence draw (*poſt.* 1.) two Right-lines CA, CB.




I ſay, $\mathfrak{AB} = \mathfrak{BC} = \mathfrak{AC}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AB = BC. *Def.* 16. |
| Alſo | 2 | AB = AC. *Def.* 16. |
| ∴ e. 1. 2. | 3 | $\mathfrak{AB} = \mathfrak{BC} = \mathfrak{AC}$. *Ax.* 1. Q.E.F. & D. |

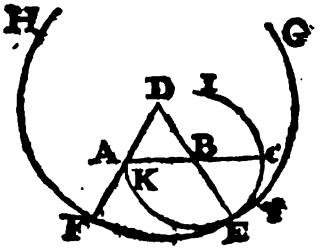
## PROP. II.

*At a Point given* A, *to make a Right-line* A F, *equal to a Right-line given* B C.

## PREPARATION.

Draw (*poſt.* 1.) the Right-line A B, and upon it deſcribe (1) an Equilateral Triangle A B D. With the Radius B C, deſcribe (*poſt.* 3.) the Arch of a Circle I C E K, and produce (*poſt.* 2.) D B to the Point E, in that Arch. With the Radius D E, deſcribe (*poſt.* 3.) the Arch of a Circle G E F H, and produce (*poſt.* 2.) D A to the Point F, in that Arch.

I ſay, 𝔄𝔉 = 𝔅𝔈.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | D F = D E. *Def.* 16. |
| | 2 | D A = D B. *Conſt.* |
| And | 2 | D F — D A = D E — D B. *Ax.* 12. |
| 1 — 2 | 3 | A F = B E. |
| i. e. | 4 | A F = B E. |
| | 5 | B C = B E. *Def.* 16. |
| Alſo | 5 | 𝔄𝔉 = 𝔅𝔈. *Ax.* 1. Q. E. F. & D. |
| ∴ a. 4. 5. | 6 | |

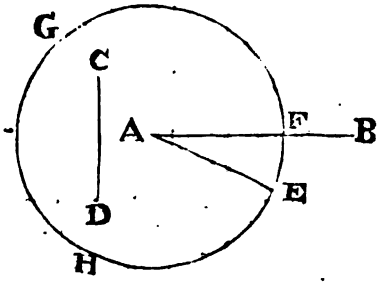
## PROP. III.

*Two unequal Right-lines being given, to cut off from the greater* A B, *a Part* A F *equal to the leſſer* C D.

PRE-

## PREPARATION.

Draw (2.) from the Point A, the Line AF = CD, and with the Radius AE, describe (*poſt.* 3.) the Circle GFEH.



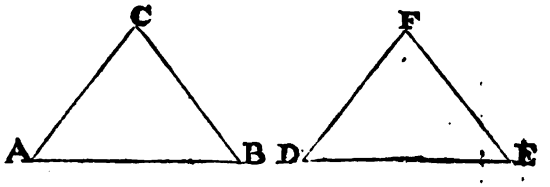

I ſay, AF = CD.

## DEMONSTRATION.

1 AF = AE. *Def.* 16.
Alſo 2 CD = AE. *Conſt.*
∴ c. 1, 2. 3 AF = CD. *Ax.* 1. Q. E. F. & D.

## PROP. IV.

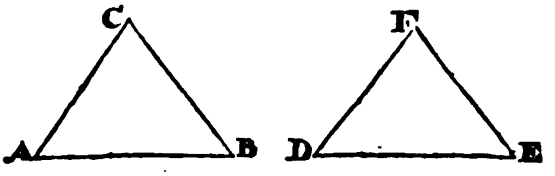
*If two Triangles* ABC, DEF, *have two Sides of the one* CA, CB, *equal to two Sides of the other* FD, FE, *each to its correſpondent Side (that is,* CA = FD, *and* CB = FE*) and have the Angle* C, *equal to the Angle* F, *comprehended between thoſe equal Sides; they ſhall have the Baſe* AB, *equal to the Baſe* DE; *and the Triangle* ABC, *ſhall be equal to the Triangle* DEF; *and the remaining Angles* A, B, *ſhall be equal to the remaining Angles* D, E, *each to each, under which the equal Sides are ſubtended.*




H That

That is, $\mathfrak{A}\mathfrak{B} = \mathfrak{D}\mathfrak{E}$. $\triangle \mathfrak{A}\mathfrak{B}\mathfrak{C} = \triangle \mathfrak{D}\mathfrak{E}\mathfrak{F}$. $\angle \mathfrak{A} = \angle \mathfrak{D}$. $\angle \mathfrak{B} = \angle \mathfrak{E}$.

## DEMONSTRATION.




| | | |
|---|---|---|
| For | 1 | If $\triangle$ ABC be placed on $\triangle$ DEF. |
| Then | 2 | CA being = FD. *Hyp.* |
| ∵ e. 2. | 3 | The Point A will fall on D, the Point C on F. |
| Alſo | 4 | $\angle$C being = $\angle$F, CB = FE. *Hyp.* |
| ∵ e. 4. | 5 | CB will fall on FE, and B on E. |
| ∵ e. 3. 5. | 6 | The Points A, C, B, will fall on D, F, E. |
| ∵ e. 6. | 7 | AB will fall on DE. *Ax.* 20. |
| ∴ | 8 | $\mathfrak{A}\mathfrak{B} = \mathfrak{D}\mathfrak{E}$. $\triangle \mathfrak{A}\mathfrak{B}\mathfrak{C} = \triangle \mathfrak{D}\mathfrak{E}\mathfrak{F}$. $\angle \mathfrak{A} = \angle \mathfrak{D}$. $\angle \mathfrak{B} = \angle \mathfrak{E}$. *Ax.* 19. Q. E. D. |

## PROP. V.

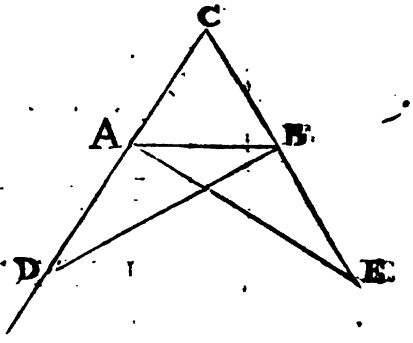
*The Angles* CAB, CBA, *at the Baſe of an Iſoſceles Triangle* ABC, *are equal one to the other: And if the equal Sides* CA, CB, *be produced, the Angles* ABE, BAD, *under the Baſe ſhall be equal one to the other.*

PRE-

## PREPARATION.

Take (3.) CD = CE; and *(post. 1.)* join BD, and AE.

I say, ∠CAB = ∠CBA. ∠ABE = ∠BAD.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CD = CE. *Const.* |
| And | 2 | CA = CB. *Const.* |
| 1 — 2 | 3 | CD — CA = CE — CB. *Ax.* 12. |
| i. e. | 4 | AD = BE. |
| Again | 5 | ∠C is common. |
| ∵ e. 2. 1. 5. | 6 | CA = CB, CE = CD, ∠C is common. |
| ∵ e. 6. | 7 | AE = BD, ∠E = ∠D, ∠CAE = ∠CBD. 4. |
| Alſo. e. 4. 7. | 8 | BE = AD, AE = BD, ∠E = ∠D. |
| ∵ e. 8. | 9 | ∠ABE = ∠BAD. 4. |
| Alſo. e. 8. | 10 | ∠BAE = ∠ABD. 4. |
| Laſtly e. 7. | 11 | ∠CAE = ∠CBD. |
| 11 — 10. | 12 | ∠CAE — ∠BAE = ∠CBD — ∠ABD. *Ax.* 12. |
| i. e. | 13 | ∠CAB = ∠CBA. |
| ∵ e. 13. 9. | 14 | ∠CAB = ∠CBA. ∠ABE = ∠BAD. *Q. E. D.* |

## COROLLARY.

Hence, Every Equilateral Triangle is alſo Equiangular.

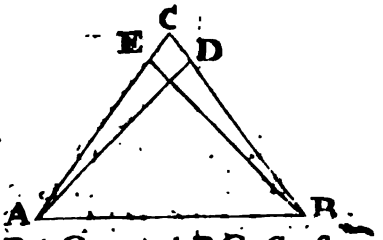
PER-

## PROP VI.

*If two Angles* ABC, BAC, *of a Triangle* ABC, *be equal, the one to the other, the Sides* AC, BC, *ſubtended under the equal Angles, ſhall alſo be equal.*

### PREPARATION.

If AC be ſaid to be greater than BC, make (3.) AE = BC, if leſs, make (3.) BD = AC, and draw (*poſt.* 1.) AD, BE.




I ſay, $\mathfrak{AC} = \mathfrak{BC}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If AC was $\sqsubset$ BC. |
| Then | 2 | AB being common, AE = BC, $\angle$BAC = $\angle$ABC. *Conſt.* |
| ∴ e. 2. | 3 | $\triangle$ ABE would be = $\triangle$ ABC. 4. |
| But | 4 | $\triangle$ ABE $\neg$ $\triangle$ ABC. *Ax.* 17. |
| ∴ | 5 | AC is not $\sqsubset$ BC. |
| Again, | 6 | If AC was $\neg$ BC. |
| Then | 7 | AB being common, BD = AC, $\angle$ABC = $\angle$BAC. *Conſt.* |
| ∴ e. 7. | 8 | $\triangle$ ABD would be = $\triangle$ ABC. 4. |
| But | 9 | $\triangle$ ABD $\neg$ $\triangle$ ABC. *Ax.* 17. |
| ∴ | 10 | AC is not $\neg$ BC. |
| ∴ e. 5. 10. | 11 | AC is neither $\sqsubset$ nor $\neg$ BC. |
| ∴ e. 11. | 12 | $\mathfrak{AC} = \mathfrak{BC}$. *Ax.* 4. *Q. E. D.* |

### COROLLARY.

Hence, every Equiangular Triangle is alſo Equilateral.

*Note.* We ſhall here omit the Seventh *Propoſition*, as only ſerving by Way of *Lemma* to the Eighth, which may be demonſtrated without it.

PROP.

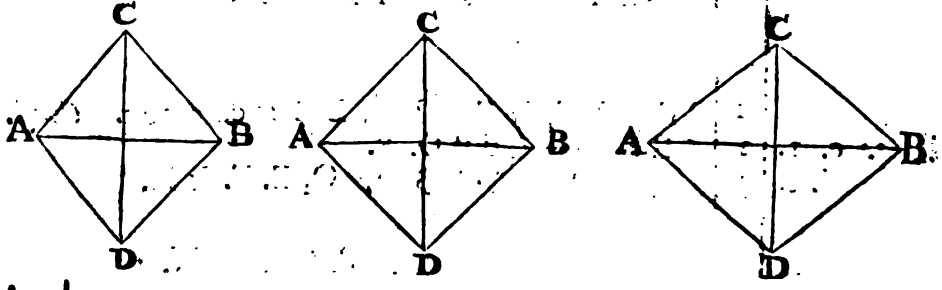
## PROP. VIII.

*If in two Triangles* A B C, A B D, *the two Sides* A C, B C, *of the one, be equal to the two Sides* A D, B D, *of the other, each to its correſpondent Side (that is,* A C = A D, *and* B C = B D) *and the Baſe* A B *be common to to them both, which is the ſame as to have equal Baſes; then the Angles contained under the equal Sides ſhall be equal, viz.* ∠ A C B = ∠ A D B, ∠ C A B = ∠ D A B, ∠ C B A = ∠ D B A.

### PREPARATION.

From the Point C, to the Point D, in the Acute, Right, and Obtuſe-Angled Triangles A B C, A B D, draw *(poſt. 1.)* the Right-line C D.

I ſay, ∠ A C B = ∠ A D B. ∠ C A B = ∠ D A B. ∠ C B A = ∠ D B A.




### DEMONSTRATION.

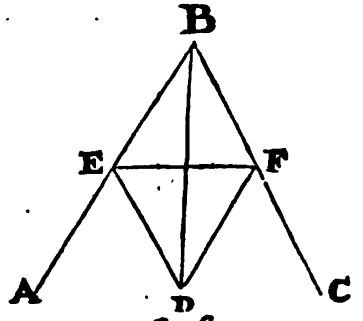
| | | |
|---|---|---|
| | 1 | A C = A D. *Hyp.* |
| ∵ e. 1. | 2 | ∠ A C D = ∠ A D C. 5. |
| Again | 3 | B C = B D. *Hyp.* |
| ∵ e. 3. | 4 | ∠ B C D = ∠ B D C. 5. |
| a + a | 5 | ∠ A C D + ∠ B C D = ∠ A D C + ∠ B D C. *Ax.* 11. |
| i. e. | 6 | ∠ A C B = ∠ A D B. |
| Laſtly, e. 1. 3. 6. | 7 | A C = A D, B C = B D, ∠ A C B = ∠ A D B. |
| ∵ e. 7. | 8 | ∠ C A B = ∠ D A B, ∠ C B A = ∠ D B A. 4. |
| ∵ e. 6. 8. | 9 | ∠ A C B = ∠ A D B. ∠ C A B = ∠ D A B. ∠ C B A = ∠ D B A. *Q. E. D.* |

## PROP. IX.

*To divide an Angle* ABC, *into two equal Parts, or Angles* ABD, CBD.

### PREPARATION.

Take (3.) BE = BF, and draw *(poſt. 1.)* the Line EF; upon which make (1.) an Equilateral Triangle EDF, and draw *(poſt. 1.)* the Right-line BD.




I ſay, ∠ABD = ∠CBD.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | BE = BF, DE = DF, BD is common. *Conſt.* |
| v. e. 1. | 2 | ∠EBD = ∠FBD. 8. |
| i. e. | 3 | ∠ABD = ∠CBD. Q. E. F. & D. |

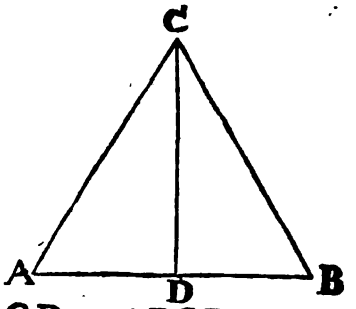
## PROP. X.

*To Biſect a Right-line given* AB.

### PREPARATION.

Upon the Line given AB, erect *(1.)* an Equilateral Triangle ABC; and *(9.)* Biſect the Angle C, with the Right-line CD.

I say, $\mathfrak{A}\mathfrak{D} = \mathfrak{D}\mathfrak{B}$.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AC = BC, DC is common, ∠ACD = ∠BCD. *Const.* |
| ∴ e. 1. | 2 | $\mathfrak{A}\mathfrak{D} = \mathfrak{D}\mathfrak{B}$. 4. Q. E. F. & D. |

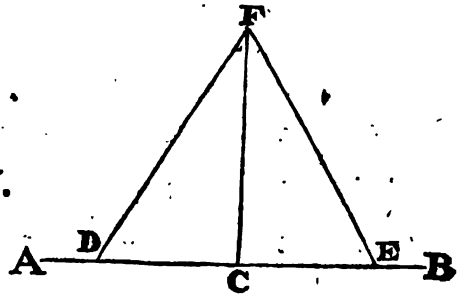
## PROP. XI.

*From a Point* C, *in a Right-line given* AB, *to erect a Perpendicular* CF.

## PREPARATION.

Take *(3.)* on either Side of the Point given, CD = CE; then upon the Right-line DE, erect *(1.)* an Equilateral Triangle DEF, and draw *(post. 1.)* the Right-line CF.

I say, $\mathfrak{C}\mathfrak{F}$ *is a* ⊥




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CD = CE, DF = EF, CF is common. *Const.* |
| ∴ e. 1. | 2 | ∠DCF = ∠ECF. 8. |
| ∴ e. 2. | 3 | $\mathfrak{C}\mathfrak{F}$ *is a* ⊥. *Def.* 10. Q. E. F. & D. |

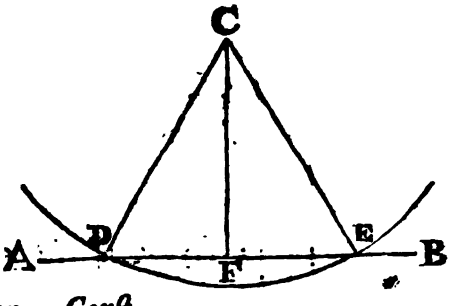
PROP.

## PROP. XII.

*Upon an Infinite Right-line given* AB, *from a Point given* C, *that is not in it, to let fall a Perpendicular Right-line* CF.

## PREPARATION.

From the given Point C, deſcribe *(poſt.* 3.*)* a Circle cutting the Right-line given AB, in the Points D and E; then (10.) biſect DE in F, and draw *(poſt.* 1.*)* the Right-lines CD, CF, CE.




I ſay, CF is a ⊥.

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CD=CE. *Def.* 16. |
| And | 2 | DF=FE, CF is common, *Conſt.* |
| ∵ e. 1. 2. | 3 | CD=CE, DF=FE, CF is common. |
| ∵ e. 3. | 4 | ∠DFC=∠EFC. 8. |
| ∵ e. 4. | 5 | CF is a ⊥. *Def.* 10. Q. E. F. & D. |

*Note.*

*Note.* In order to ſhorten ſeveral of the following Demonſtrations, it will be neceſſary to premiſe thus much.

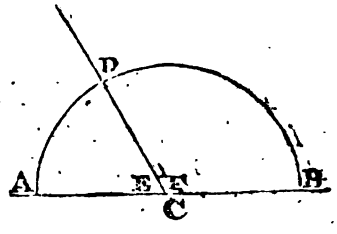
1ſt That the Periphery (or Circumference) of every Circle (whether Great or Small) is ſuppoſed to be divided into 360 equal Parts, called *Degrees.*

2dly That all Angles are meaſured by the Arch of a Circle deſcribed upon the Angular Point, and are eſteemed Greater or Leſs, according to the Number of Degrees contained in that Arch.

3dly That a *Quadrant*, or *Quarter Part of any Circle*, is always = 90° Degrees, being the Meaſure of a Right-angle; and that a Semi-circle is = 180° Degrees, being the Meaſure of two Right-angles; alſo that equal Arches of a Circle, or of equal Circles, meaſure equal Angles.

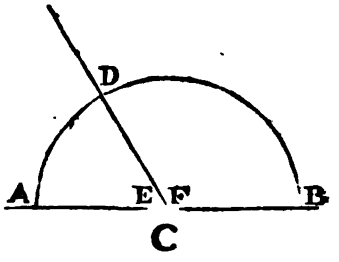
## PROP. XIII.

*If one Right-line* DC, *fall on another Right-line* AB, *it will make with it two Right-angles, or two Angles equal to two Right-angles.*



## PREPARATION.

If the Angles E, F, be equal, then (*Def.* 10.) they make two Right-angles; if unequal, then from the Point C, deſcribe any Semicircle.




I ſay, $\angle E + \angle F = 2 r\angle^s$.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | Arch AD = ∠E. |
| And | 2 | Arch DB = ∠F. |
| 1 + 2 | 3 | Arch AD + arch DB = ∠E + ∠F. *Ax.* 11. |
| Alſo | 4 | Arch AD + arch DB = $2 r\angle^s$. |
| v. e. 3. 4. | 5 | $\angle E + \angle F = 2 r\angle^s$. *Ax.* 1. Q. E. D. |

## COROLLARIES.

1. Hence, If ∠F = 90°. then ∠E = 90°. but if ∠F be Obtuſe, or ⊏ 90°. then ∠E will be Acute, or ⌝ 90°.

2. If ſeveral Right-lines ſtand upon, or meet with any Right-line at one and the ſame Point, as C, all the Angles taken together will be = $2 r\angle^s$, *viz.* 180°.

## PROP. XIV.

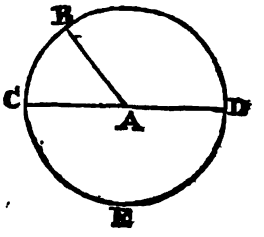
*If two Lines* CA, DA, *meeting together at the ſame Point* A, *of another Line* AB, *make with it two Angles* CAB, BAD, *equal to two Right ones; they will make one ſtraight Line.*

PRE-

## PREPARATION.

From the Center A, describe a Circle CBDE.

I say, CAD is a straight Line.



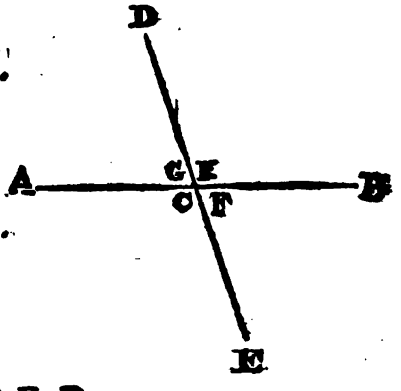

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\angle CAB + \angle BAD = 2r\angle^s$. *Hyp.* |
| ∵ e. 1. | 2 | Arch CBD is a Semicircle. |
| ∵ e. 2. | 3 | CAD is the Diameter. |
| ∵ e. 3. | 4 | CAD is a straight Line. *Def.* 17. Q. E. D. |

## PROP. XV.

*If two Lines* AB, DE, *intersect or cross each other, the two opposite Angles at the Vertex will be equal.*

That is, $\angle G = \angle F$. $\angle K = \angle C$.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\angle G + \angle C = 2r\angle^s$. 13. |
| Also | 2 | $\angle F + \angle C = 2r\angle^s$. 13. |
| ∵ e. 1. 2. | 3 | $\angle G + \angle C = \angle F + \angle C$. *Ax.* 1. |
| 3 — $\angle C$ | 4 | $\angle G = \angle F$. *Ax.* 12. |
| Again | 5 | $\angle K + \angle F = 2r\angle^s$. 13. |
| ∵ e. 5. 2. | 6 | $\angle K + \angle F = \angle F + \angle C$. *Ax.* 1. |
| 6 — $\angle F$ | 7 | $\angle K = \angle C$. *Ax.* 12. |
| ∵ e. 4. 7. | 8 | $\angle G = \angle F$. $\angle K = \angle C$. Q. E. D. |

## COROLLARIES.

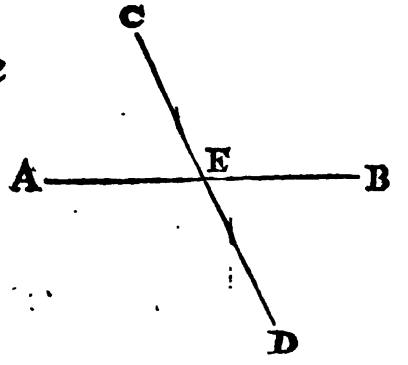
1 Hence, Two Right-lines croſſing each other make Angles equal to four Right.

2 All the Angles made about one Point make four Right.

## SCHOLIUM.

If four Right-lines E A, E B, E C, E D, proceeding from one Point E, make the Angles vertically oppoſite equal the one to the other, each two Lines, A E, E B, and C E, E D, are placed in one ſtraight Line.

That is, A E B, C E D, are ſtraight Lines.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\angle AEC = \angle BED$. *Hyp.* |
| And | 2 | $\angle CEB = \angle DEA$. *Hyp.* |
| 1 + 2 | 3 | $\angle AEC + \angle CEB = \angle BED + \angle DEA$, *Ax.* 11. |
| Again | 4 | $\angle AEC + \angle CEB + \angle BED + \angle DEA = 4r\angle^s$. *Cor.* 2. 15. |
| 4 − 3 | 5 | $\angle BED + \angle DEA = 4r\angle^s - \angle BED - \angle DEA$. *Ax.* 12. |
| 5 + ∠BED | 6 | $2\angle BED + \angle DEA = 4r\angle^s - \angle DEA$. *Ax.* 11. |
| 6 + ∠DEA | 7 | $2\angle BED + 2\angle DEA = 4r\angle^s$. *Ax.* 11. |
| 7 ÷ 2 | 8 | $\angle BED + \angle DEA = 2r\angle^s$. *Ax.* 14. |
| ∵ e. 8. | 9 | A E B is a ſtraight Line. 14. |
| Laſtly, e. 8. 2. | 10 | $\angle BED + \angle CEB = 2r\angle^s$. *Sub.* |
| ∵ e. 10. | 11 | C E D is a ſtraight Line. 14. |
| ∵ e. 9. 11. | 12 | A E B, C E D, are ſtraight Lines. Q. E. D. |

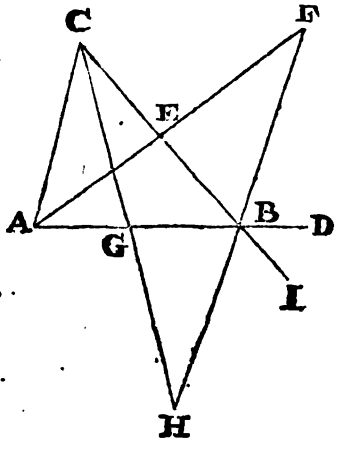
PROP.

## PROP. XVI.

*AB, one of the Sides of a Triangle ABC, being produced, the exterior Angle CBD, is greater than either of the interior or oppoſite Angles ACB, CAB.*

## PREPARATION.

Make (10.) CE = EB, AG = GB, and drawing the Lines AE, CG, and extending them to F and H, ſo that AE be = EF, and CG = GH, join BF, BH, and extend CB to I.




I ſay, ∠CBD ⊏ ∠ACB or ∠CAB.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AE = EF, CE = EB. *Conſt.* |
| And | 2 | ∠AEC = ∠BEF. 15. |
| ∵ e. 1. 2. | 3 | AE = EF, CE = EB, ∠AEC = ∠BEF. |
| ∵ e. 3. | 4 | ∠ACE = ∠EBF. 4. |
| But | 5 | ∠CBD ⊏ ∠EBF. *Ax.* 17. |
| ∵ e. 4. 5. | 6 | ∠CBD ⊏ ∠ACE. *Ax.* 2. |
| i. e. | 7 | ∠CBD ⊏ ∠ACB. |
| Again | 8 | CG = GH, AG = GB. *Conſt.* |
| And | 9 | ∠AGC = ∠BGH. 15. |
| ∵ e. 8. 9. | 10 | CG = GH, AG = GB, ∠AGC = ∠BGH. |
| ∵ e. 10. | 11 | ∠CAG = ∠GBH. 4. |
| But | 12 | ∠GBI ⊏ ∠GBH. *Ax.* 17. |
| ∵ e. 11. 12. | 13 | ∠GBI ⊏ ∠CAG. *Ax.* 2. |
| Now. | 14 | ∠GBI = ∠CBD. 15. |
| ∵ e. 13. 14. | 15 | ∠CBD ⊏ ∠CAG. *Ax.* 3. |
| i. e. | 16 | ∠CBD ⊏ ∠CAB. |
| ∵ e. 7. 16. | 17 | ∠CBD ⊏ ∠ACB or ∠CAB. Q. E. D. |

## PROP. XVII.

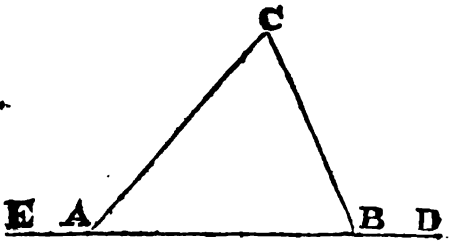
*Two Angles of any Triangle* ABC, *which way ſoever they are taken, are leſs than two Right-angles.*

## PREPARATION.

Produce the Side AB both ways.

I ſay, $\angle ABC + \angle BAC$ or $\angle BAC + \angle C$ or $\angle C + \angle ABC \sqcap 2r\angle^s$.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $\angle BAC \sqcap \angle CBD$. 16. |
| 1 + ∠ABC | 2 | $\angle ABC + \angle BAC \sqcap \angle ABC + \angle CBD$. *Ax.* 15. |
| But | 3 | $\angle ABC + \angle CBD = 2r\angle^s$. 13. |
| ∵ e. 2. 3. | 4 | $\angle ABC + \angle BAC \sqcap 2r\angle^s$. *Ax.* 2. |
| Again | 5 | $\angle C \sqcap \angle CAE$. 16. |
| 5 + ∠BAC | 6 | $\angle BAC + \angle C \sqcap \angle BAC + \angle CAE$. *Ax.* 15. |
| But | 7 | $\angle BAC + \angle CAE = 2r\angle^s$. 13. |
| ∵ e. 6. 7. | 8 | $\angle BAC + \angle C \sqcap 2r\angle^s$. *Ax.* 2 |
| Laſtly | 9 | $\angle C \sqcap \angle CBD$. 16. |
| 9 + ∠ABC | 10 | $\angle C + \angle ABC \sqcap \angle CBD + \angle ABC$. *Ax.* 15. |
| But | 11 | $\angle CBD + \angle ABC = 2r\angle^s$. 13. |
| ∵ e. 10. 11. | 12 | $\angle C + \angle ABC \sqcap 2r\angle^s$. *Ax.* 2. |
| ∵ e. 4. 8. 12. | 13 | $\angle ABC + \angle BAC$ or $\angle BAC + \angle C$ or $\angle C + \angle ABC \sqcap 2r\angle^s$. Q. E. D. |

COROL.

## COROLLARIES.

1. Hence, In every Triangle, wherein One Angle is either Right or Obtuſe, the two others are Acute Angles.

2. All the Angles of an Equilateral Triangle, and the two Angles of an Iſoſceles Triangle that are upon the Baſe, are Acute.

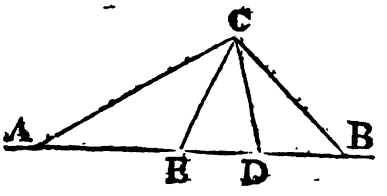
## PROP. XVIII.

*The greateſt Side* AB, *of every Triangle* ABC, *ſubtends the greateſt Angle* ACB.

## PREPARATION.

From AB, take (3.) AD = AC, BE = BC, and join CD, CE.

I ſay, ∠ACB ⊏ ∠B or ∠A.



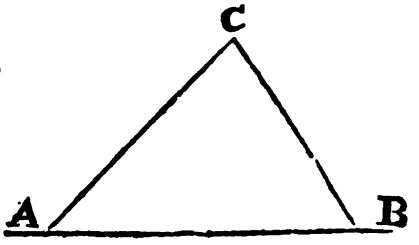

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AD = AC. *Conſt.* |
| ∵ e. 1. | 2 | ∠ADC = ∠ACD. 5. |
| Alſo | 3 | ∠ADC ⊏ ∠B. 16. |
| ∵ e. 2. 3. | 4 | ∠ACD ⊏ ∠B. *Ax.* 3. |
| But | 5 | ∠ACB ⊏ ∠ACD. *Ax.* 17. |
| ∵ e. 4. 5. | 6 | ∠ACB is much ⊏ ∠B. |
| Again | 7 | BE = BC. *Conſt.* |
| ∵ e. 7. | 8 | ∠BEC = ∠BCE. 5. |
| Alſo | 9 | ∠BEC ⊏ ∠A. 16. |
| ∵ e. 8. 9. | 10 | ∠BCE ⊏ ∠A. *Ax.* 3. |
| But | 11 | ∠ACB ⊏ ∠BCE. *Ax.* 17. |
| ∵ e. 10. 11. | 12 | ∠ACB is much ⊏ ∠A. |
| ∵ e. 6. 12. | 13 | ∠ACB ⊏ ∠B or ∠A. Q. E. D. |

## PROP. XIX.

*In every Triangle* ABC, *under the greateſt Angle* C, *is ſubtended the greateſt Side* AB.

That is, 𝔄𝔅 ⊏ 𝔄ℭ or 𝔅ℭ.




### DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If AB was = AC. |
| Then | 2 | ∠B would be = ∠C. 5. |
| But | 3 | ∠B ⌝ ∠C. *Hyp.* |
| ∴ | 4 | AB is not = AC. |
| Alſo | 5 | If AB was ⌝ AC. |
| Then | 6 | ∠B would be ⊏ ∠C. 18. |
| But | 7 | ∠B ⌝ ∠C. *Hyp.* |
| ∴ | 8 | AB is not ⌝ AC. |
| ∴ e. 4. 8. | 9 | AB is neither = nor ⌝ AC. |
| ∴ e. 9. | 10 | AB ⊏ AC. *Ax.* 5. |
| Again | 11 | If AB was = BC. |
| Then | 12 | ∠A would be = ∠C. 5. |
| But | 13 | ∠A ⌝ ∠C. *Hyp.* |
| ∴ | 14 | AB is not = BC. |
| Alſo | 15 | If AB was ⌝ BC. |
| Then | 16 | ∠A would be ⊏ ∠C. 18. |
| But | 17 | ∠A ⌝ ∠C. *Hyp.* |
| ∴ | 18 | AB is not ⌝ BC. |
| ∴ e. 14. 18. | 19 | AB is neither = nor ⌝ BC. |
| ∴ e. 19. | 20 | AB ⊏ BC. *Ax.* 5. |
| ∴ e. 10. 20. | 21 | 𝔄𝔅 ⊏ 𝔄ℭ or 𝔅ℭ. Q. E. D. |

PROP.

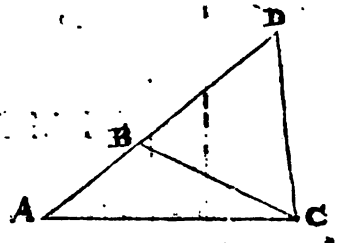
## PROP. XX.

*Of any Triangle* ABC, *two Sides any way taken, are greater than that which remains; so that the Sum of the leſſer Sides* AB, BC, *is greater than the greateſt* AC.

### PREPARATION.

Produce AB; and taking (3.) BD = BC, join DC.

I ſay, $\mathfrak{AB}+\mathfrak{BC}\sqsubset\mathfrak{AC}$.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | BD = BC. *Conſt.* |
| ∵ e. 1. | 2 | ∠D = ∠BCD. 5. |
| But | 3 | ∠ACD ⊏ ∠BCD. *Ax.* 17. |
| ∵ e. 2. 3. | 4 | ∠ACD ⊏ ∠D. *Ax.* 2. |
| ∵ e. 4. | 5 | AD ⊏ AC. 19. |
| i. e. | 6 | AB + BD ⊏ AC. |
| ie. e. 1. 6. | 7 | $\mathfrak{AB}+\mathfrak{BC}\sqsubset\mathfrak{AC}$. *Sub.* Q. E. D. |

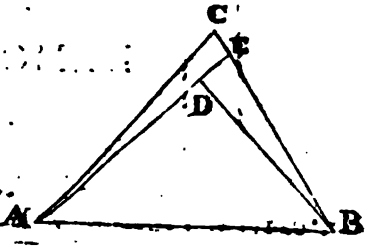
## PROP. XXI.

*If from the Point* D, *taken at Pleasure in any Triangle* ABC, *the Right-lines* DB, DA, *be drawn to the Extreams* B, A, *of one of its Sides* AB, *their Sum* DB + DA, *will be less than the Sum* AC + CB, *of the other two Sides* AC, CB; *but the Angle* ADB, *will be greater than the Angle* ACB.

### PREPARATION.

Extend DA to E.

I say, DB + DA ⌝ AC + CB. ∠ADB ⌐ ACB.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AE ⌝ AC + CE. 20. |
| 1 + EB. | 2 | AE + EB ⌝ AC + CE + EB. Ax. 3. |
| i.e. | 3 | AE + EB ⌝ AC + CB. |
| Again | 4 | DB ⌝ EB + ED. 20. |
| 4 + DA. | 5 | DB + DA ⌝ EB + ED + DA. Ax. 1. |
| i.e. | 6 | DB + DA ⌝ AE + EB. |
| But c. 3. | 7 | AE + EB ⌝ AC + CB. |
| ∴ c. 6. 7. | 8 | DB + DA is much ⌝ AC + CB. |
| Lastly | 9 | ∠ADB ⌐ ∠DEB. 16. |
| Also | 10 | ∠DEB ⌐ ∠ACE. 16. |
| ∴ c. 9. 10. | 11 | ∠ADB is much ⌐ ∠ACE. |
| i.e. | 12 | ∠ADB is much ⌐ ∠ACB. |
| ∴ c. 8. 12. | 1, | DB + DA ⌝ AC + CB. ∠ADB ⌐ ACB. Q. E. D. |

PROP.

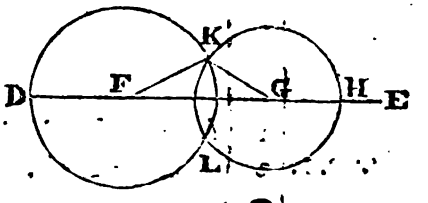
## PROP. XXII.

*Of three given Right-lines* A, B, C, *to make a Triangle* K F G, *whereof the biggeſt ought* (20.) *to be leſs than the Sum of the other two.*

## PREPARATION.

Draw the Infinite Right-line D E, and from it take (3.) F D = A, F G = B, G H = C; then from the Centers F and G, with the Radij F D, G H, deſcribe (*Poſt.* 3.) the Circles D K L, H K L, cutting each other in the Point K, and from that Point draw K F, K G.

I ſay, K F G is the △ ſought.




A —— B —— C

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | F D = K F. *Def.* 16. |
| Alſo | 2 | F D = A. *Conſt.* |
| ∴ e. 1. 2. | 3 | K F = A. *Ax.* 1. |
| Again | 4 | F G = B. *Conſt.* |
| Laſtly | 5 | G H = K G. *Def.* 16. |
| Alſo | 6 | G H = C. *Conſt.* |
| ∴ e. 5. 6. | 7 | K G = C. *Ax.* 1. |
| ∴ e. 3. 4. 7. | 8 | K F = A, F G = B, K G = C. |
| ∴ e. 8. | 9 | K F G is the △ ſought. Q. E. F. & D. |

## PROP. XXIII.

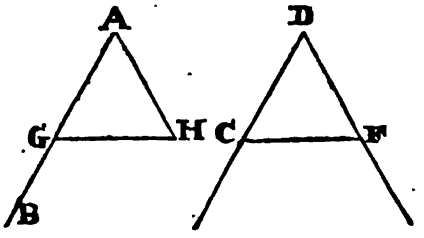
*At a Point* A, *in a Right-line given* AB, *to make a Right-lined Angle* A, *equal to a Right-lined Angle given* D.

### PREPARATION.

Draw the Right-line CF, cutting the Sides of the Angle given any ways; make (3.) AG = DC; upon AG, raiſe (22.) a Triangle AGH, Equilateral to the former DCF, ſo that AH be = DF, and GH = CF.

I ſay, ∠A = ∠D.

### DEMONSRATION.




1 AG = DC, AH = DF, GH = CF. *Conſt.*
∵ c. 1. 2 ∠A = ∠D. 8. Q. E. F. & D.

## PROP. XXIV.

*If two Triangles* ABC, DEF, *have two Sides of the one Triangle* AB, AC, *equal to two Sides of the other Triangle* DE, DF, *each to its correſpondent Side, (that is,* AB = DE, *and* AC = DF,) *and have the Angle* A *greater than the Angle* EDF, *comprehended under the equal Right-lines, they ſhall alſo have the Baſe* BC *greater than the Baſe* EF.

PRE-

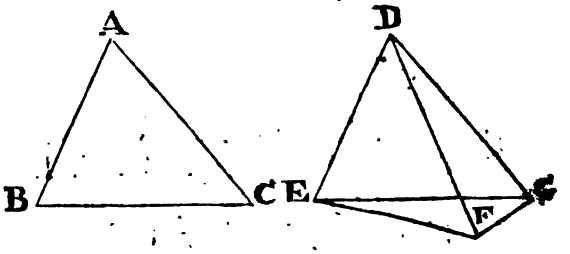
## PREPARATION.

Make (23.) $\angle EDG = \angle A$, and (3.) $DG = DF =$ *(Hyp.)* AC, and join EG, FG.

*Case.* 1 If EG fall above EF.

I say, BC ⊏ EF.

## DEMONSTRATION.

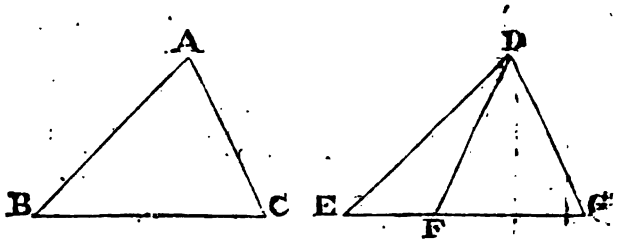



| | | |
|---|---|---|
| | 1 | $AB = DE$. *Hyp.* |
| And | 2 | $AC = DG$, $\angle A = \angle EDG$. *Const.* |
| ∴ e. 1. 2. | 3 | $AB = DE$, $AC = DG$, $\angle A = \angle EDG$. |
| ∴ e. 3. | 4 | $BC = EG$. 4. |
| Again | 5 | $DG = DF$. *Const.* |
| ∴ e. 5. | 6 | $\angle DGF = \angle DFG$. 5. |
| But | 7 | $\angle DGF$ ⊏ $\angle EGF$. *Ax.* 17. |
| ∴ e. 6. 7. | 8 | $\angle DFG$ ⊏ $\angle EGF$. *Ax.* 3. |
| Now | 9 | $\angle EFG$ ⊏ $\angle DFG$. *Ax.* 17. |
| ∴ e. 8. 9. | 10 | $\angle EFG$ is much ⊏ $\angle EGF$. |
| ∴ e. 10. | 11 | EG ⊏ EF. 19. |
| But e. 4. | 12 | $EG = BC$. |
| ∴ e. 11. 12. | 13 | BC ⊏ EF. *Ax.* 3. Q. E. D. |

*Case.* 2 If EG fall upon EF.

I say, ∠BC ⊏ ∠EF.

DEMONSTRATION.

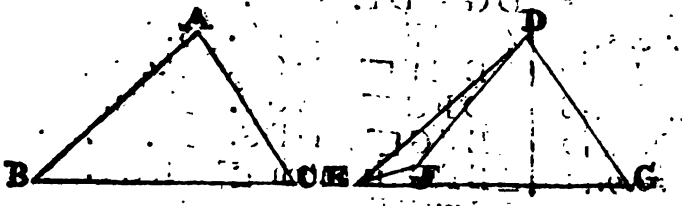



1 EG ⊏ EF. *Ax.* 17.
But 2 EG = BC. *Step.* 4. *Case* 1.
∴ e. 1. 2. 3 ∠BC ⊏ ∠EF. *Ax.* 3. Q. E. D.

*Case.* 3 If EG fall below EF.

I say ∠BC ⊏ ∠EF.

DEMONSTRATION.




1 DG + EG ⊏ DF + EF. 21.
And 2 DG = DF. *Const.*
∴ e. 1. 2. 3 DG + EG ⊏ DG + EF. *Sub.*
3 — DG 4 EG ⊏ EF. *Ax.* 16.
But 5 EG = BC. *Step* 4. *Case* 1.
∴ e. 4. 5. 6 ∠BC ⊏ ∠EF. *Ax.* 3. Q. E. D.

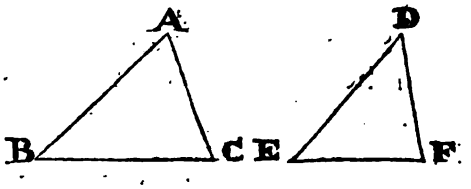
PROP.

## PROP. XXV.

*If two Triangles* ABC, DEF, *have two Sides* AB, AC, *equal to two Sides* DE, DF, *each to its correſpondent Side, (that is,* AB = DE, *and* AC = DF*) and have the Baſe* BC *greater than the Baſe* EF, *they ſhall alſo have the Angle* A, *comprehended under the equal Right-lines, greater than the Angle* D.

That is, ∠A ⊏ ∠D.

## DEMONSTRATION.




| | | |
|---|---|---|
| For | 1 | If ∠A was = ∠D. |
| Then | 2 | AB being = DE, AC = DF, ∠A = ∠D. *Hyp.* |
| ∴ e. 2. | 3 | BC would be = EF. 4. |
| But | 4 | BC ⊏ EF. *Hyp.* |
| ∴ | 5 | ∠A is not = ∠D. |
| Again | 6 | If ∠A was ⊐ ∠D. |
| Then | 7 | BC would be ⊐ EF. 24. |
| But | 8 | BC ⊏ EF. *Hyp.* |
| ∴ | 9 | ∠A is not ⊐ ∠D. |
| ∴ e. 5. 9 | 10 | ∠A is neither = nor ⊐ ∠D. |
| ∴ e. 10. | 11 | ∠A ⊏ ∠D. *Ax.* 5. Q. E. D. |

## PROP. XXVI.

*If two Triangles* ABC, DEF, *have two Angles of the one,* A, ABC, *equal to two Angles of the other* D, DEF, *each to its correſpondent Angle, (that is,* ∠A = ∠D, *and* ∠ABC = ∠DEF) *and have alſo one Side of the one, equal to one Side of the other, either that Side which is comprehended betwixt the equal Angles, or that which is ſubtended under one of the equal Angles; the other Sides alſo of the one, ſhall be equal to the other Sides of the other, each to its correſpondent Side, and the other Angle of the one, ſhall be equal to the other Angle of the other, and the two Triangles entirely equal.*

*Hyp.* 1 Suppoſe AB = DE, which are the Sides comprehended betwixt the equal Angles.

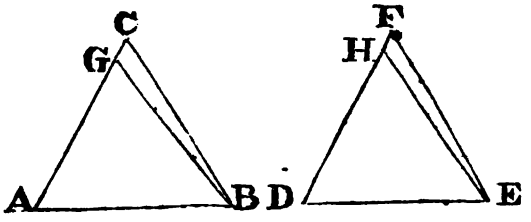
## PREPARATION.

If AC be ſaid to be greater than DF, make (3.) AG = DF, if leſs, make (3.) DH = AC, and draw BG, EH.

I ſay, 𝔄𝔊 = 𝔇𝔉. 𝔅𝔊 = 𝔈𝔉. ∠𝔄𝔊𝔅 = ∠𝔇𝔉𝔈, △𝔄𝔅𝔊 = △𝔇𝔈𝔉.

DEMON-

## DEMONSTRATION.




| | | |
|---|---|---|
| For | 1 | If AC was ⌐ DF. |
| Then | 2 | AG being = DF, AB = DE, ∠A = ∠D. *Hyp.* |
| ∵ e. 2 | 3 | ∠ABG would be = ∠DEF. 4. |
| Alſo | 4 | ∠ABC = ∠DEF. *Hyp.* |
| ∵ e. 3. 4. | 5 | ∠ABG would be = ∠ABC. *Ax.* 1. |
| But | 6 | ∠ABG ⌝ ∠ABC. *Ax.* 17. |
| ∵ | 7 | AC is not ⌐ DF. |
| Again | 8 | If AC was ⌝ DF. |
| Then | 9 | AC being = DH, AB = DE, ∠A = ∠D. *Hyp.* |
| ∵ e. 9. | 10 | ∠ABC would be = ∠DEH. 4. |
| Alſo | 11 | ∠ABC = ∠DEF. *Hyp.* |
| ∵ e. 10. 11. | 12 | ∠DEH would be = ∠DEF. *Ax.* 1. |
| But | 13 | ∠DEH ⌝ ∠DEF. *Ax.* 17. |
| ∵ | 14 | AC is not ⌝ DF. |
| ∵ e. 7. 14. | 15 | AC is neither ⌐ nor ⌝ DF. |
| ∵ e. 15. | 16 | AC = DF. *Ax.* 4. |
| ∵ e. 16. 2. | 17 | AC = DF, AB = DE, ∠A = ∠D. |
| ∵ e. 17. | 18 | 𝔄𝔊 = 𝔇𝔉. 𝔅𝔊 = 𝔈𝔉. ∠𝔄𝔊𝔅 = ∠𝔇𝔉𝔈. △𝔄𝔅𝔊 = △𝔇𝔈𝔉. 4. Q. E. D. |

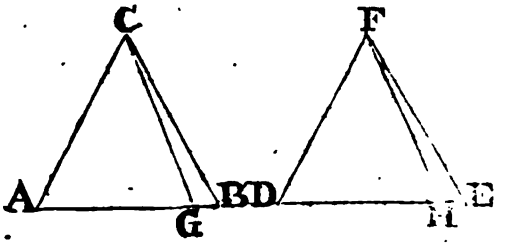
| | | |
|---|---|---|
| *Hyp.* | 2 | Suppoſe AC = DF, which are the Sides ſubtended under the equal Angles. |

PREPARATION.

If AB be ſaid to be greater than DE, make (3.) AG = DE, if leſs, make (3.) DH = AB, and draw CG, FH.

I ſay, AB = DE. BC = EF. ∠ACB = ∠DFE. △ABC = △DEF.

DEMONSTRATION.




| | | |
|---|---|---|
| For | 1 | If AB was ⊏ DE. |
| Then | 2 | AG being = DE, AC = DF, ∠A = ∠D. *Hyp.* |
| ∵ e. 2. | 3 | ∠AGC would be = ∠DEF. 4. |
| Alſo | 4 | ∠ABC = ∠DEF. *Hyp.* |
| ∵ e. 3. 4. | 5 | ∠AGC would be = ∠ABC. *Ax.* 1. |
| But | 6 | ∠AGC ⊏ ∠ABC. 16. |
| ∵ | 7 | AB is not ⊏ DE. |
| Again | 8 | If AB was ⊐ DE. |
| Then | 9 | AB being = DH, AC = DF, ∠A = ∠D. *Hyp.* |
| ∵ e. 9. | 10 | ∠ABC would be = ∠DHF. 4. |
| Alſo | 11 | ∠ABC = ∠DEF. *Hyp.* |
| ∵ e. 10. 11. | 12 | ∠DHF would be = ∠DEF. *Ax.* 1. |
| But | 13 | ∠DHF ⊏ ∠DEF. 16. |
| ∵ | 14 | AB is not ⊐ DE. |
| ∵ e. 7. 14. | 15 | AB is neither ⊏ nor ⊐ DE. |
| ∵ e. 15. | 16 | AB = DE. *Ax.* 4. |
| ∵ e. 16. 2. | 17 | AB = DE, AC = DF, ∠A = ∠D. |
| ∵ e. 17. | 18 | AB = DE. BC = EF. ∠ACB = ∠DFE. △ABC = △DEF. 4. Q. E. D. |

PROP.

## PROP. XXVII.

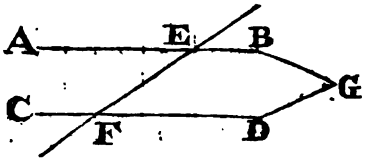
*If a Right-line* EF, *falling upon two Right-lines* AB, CD, *make the alternate Angles* AEF, DFE, *equal the one to the other, then are the Right-lines* AB, CD, *parallel.*

## PREPARATION.

If AB, CD, be ſaid not to be parallel, ſuppoſe them to be produced 'till they meet in ſome Point G, ſo will EGF be a Triangle.

I ſay, $\mathfrak{AB} \parallel \mathfrak{CD}$.

## DEMONSTRATION.



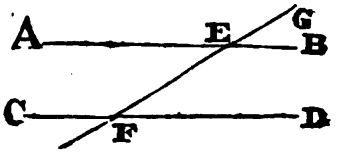

For 1 If AB, CD, being produced, ſhould ever meet; as in G.
Then 2 $\angle AEF$ would be $\sqsubset \angle DFE$. 16.
But 3 $\angle AEF = \angle DFE$. *Hyp.*
$\therefore$ 4 AB, CD, being produced, will never meet; as in G.
$\therefore$ 5 $\mathfrak{AB} \parallel \mathfrak{CD}$. *Def.* 34. Q. E. D.

## PROP. XXVIII.

*If one Right-line* GF, *cut two other Right-lines* AB, CD, *ſo that the exterior Angle* BEG, *be equal to the interior and oppoſite on the ſame Side* DFE, *or that the two interior on the ſame Side* BEF, DFE, *be together equal to two right Ones, thoſe two Right-lines will be parallel.*

*Hyp.* 1 Suppoſe $\angle BEG = \angle DFE$.

I ſay, $\mathfrak{AB} \parallel \mathfrak{CD}$.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\angle BEG = \angle AEF$. 15. |
| Alſo | 2 | $\angle BEG = \angle DFE$. *Hyp.* |
| ∵ e. 1. 2. | 3 | $\angle AEF = \angle DFE$. *Ax.* 1. |
| ∵ e. 3. | 4 | $\mathfrak{AB} \parallel \mathfrak{CD}$. 27. Q. E. D. |

*Hyp.* 2 Suppoſe $\angle BEF + \angle DFE = 2$ r$\angle$s.

I ſay, $\mathfrak{AB} \parallel \mathfrak{CD}$.

### DEMONSTRATION.

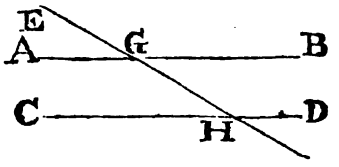
| | | |
|---|---|---|
| | 1 | $\angle BEF + \angle AEF = 2$ r$\angle$s. 13. |
| Alſo | 2 | $\angle BEF + \angle DFE = 2$ r$\angle$s. *Hyp.* |
| ∵ e. 1. 2. | 3 | $\angle BEF + \angle AEF = \angle BEF + \angle DFE$. *Ax.* 1. |
| 3 — $\angle BEF$ | 4 | $\angle AEF = \angle DFE$. *Ax.* 12. |
| ∵ e. 4. | 5 | $\mathfrak{AB} \parallel \mathfrak{CD}$. 27. Q. E. D. |

PROP.

## PROP. XXIX.

*If a Right-line* EH, *interſect two Parallels* AB, CD, *the alternate Angles* AGH, GHD, *will be equal to one another; the exterior Angle* EGB, *will be equal to the interior and oppoſite on the ſame Side* GHD, *and the two interiors on the ſame Side* BGH, GHD, *will be together equal to two Right-angles.*

That is, ∠AGH = ∠GHD. ∠EGB = ∠GHD. ∠BGH + ∠GHD = 2 r∠s.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AB ‖ CD. *Hyp.* |
| ∵ e. 1. | 2 | ∠BGH + ∠GHD = 2 r∠s. *Ax.* 22. |
| Alſo | 3 | ∠BGH + ∠AGH = 2 r∠s. 13. |
| ∵ e. 3. 2. | 4 | ∠BGH + ∠AGH = ∠BGH + ∠GHD. *Ax.* 1. |
| 4 − ∠BGH | 5 | ∠AGH = ∠GHD. *Ax.* 12. |
| Alſo | 6 | ∠AGH = ∠EGB. 15. |
| ∵ e. 6. 5. | 7 | ∠EGB = ∠GHD. *Ax.* 1. |
| ∵ e. 5. 7. 2. | 8 | ∠AGH = ∠GHD. ∠EGB = ∠GHD. ∠BGH + ∠GHD = 2 r∠s. Q. E. D. |

## PROP. XXX.

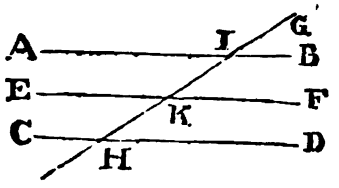
*Right-lines* AB, CD, *which are parallel to one and the same Right-line* EF, *are parallel to each other.*

### PREPARATION.

Draw GH cutting the three Right-lines any ways.

I say, AB ‖ CD.

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AB ‖ EF. *Hyp.* |
| ∵ e. 1. | 2 | ∠GIB = ∠IKF. 29. |
| Again | 3 | CD ‖ EF. *Hyp.* |
| ∵ e. 3. | 4 | ∠KHD = ∠IKF. 29. |
| ∵ e. 2. 4. | 5 | ∠GIB = ∠KHD. *Ax.* 1. |
| i. e. | 6 | ∠GIB = ∠IHD. |
| ∵ e. 6. | 7 | AB ‖ CD. 28. Q. E. D. |

## PROP. XXXI.

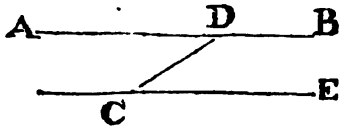
*To draw thro' a given Point* C, *a Right-line parallel to a given Right-line* AB.

### PREPARATION.

*From the given Point* C, *draw a Right-line* CD, *to any Point of the given Right-line* AB; *then make* (23.) *at the Point* C, *the Angle* DCE *equal to the Angle* ADC.

I say, CE ‖ AB.

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | ∠DCE = ∠ADC. *Const.* |
| ∵ e. 1. | 2 | CE ‖ AB. 27. Q. E. F. & D. |

PROP.

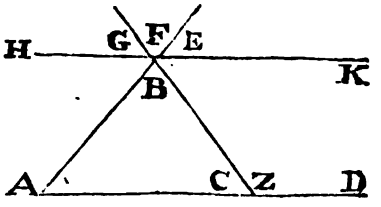
## PROP. XXXII.

*In any Triangle* ABC, *the* $\angle A + \angle B + \angle C$, viz. *the three Angles taken together, are equal to two Right-angles; and* AC *one of the Sides, being produced, the exterior* $\angle Z = \angle A + \angle B$, viz. *the two interior and oppoſite Ones taken together.*

## PREPARATION.

Draw (31.) HK || AD, juſt touching the Vertical Angle B; then produce the Sides AB, CB.

I ſay, $\angle \mathfrak{A} + \angle \mathfrak{B} + \angle \mathfrak{C} = 2\mathfrak{r}\angle^{s}$. $\angle \mathfrak{Z} = \angle \mathfrak{A} + \angle \mathfrak{B}$.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\angle E = \angle A$. 29. |
| And | 2 | $\angle F = \angle B$. 15. |
| And | 3 | $\angle G = \angle C$. 29. |
| 1+2+3 | 4 | $\angle E + \angle F + \angle G = \angle A + \angle B + \angle C$. *Ax.* 11. |
| Alſo | 5 | $\angle E + \angle F + \angle G = 2\mathfrak{r}\angle^{s}$. *Cor.* 2. 13. |
| ∴ e. 4. 5. | 6 | $\angle A + \angle B + \angle C = 2\mathfrak{r}\angle^{s}$. *Ax.* 1. |
| Again. | 7 | $\angle C + \angle Z = 2\mathfrak{r}\angle^{s}$. 13. |
| ∴ e. 7. 6. | 8 | $\angle C + \angle Z = \angle A + \angle B + \angle C$. *Ax.* 1. |
| 8 — ∠C. | 9 | $\angle Z = \angle A + \angle B$. *Ax.* 12. |
| ∴ e. 6. 9. | 10 | $\angle \mathfrak{A} + \angle \mathfrak{B} + \angle \mathfrak{C} = 2\mathfrak{r}\angle^{s}$. $\angle \mathfrak{Z} = \angle \mathfrak{A} + \angle \mathfrak{B}$. |

Q. E. D.

## COROLLARIES.

1 Becauſe the $3\angle^s$ of every $\triangle$ are $= 2\,r\angle^s$, therefore the $3\angle^s$ of one $\triangle$, are $=$ the $3\angle^s$ of another $\triangle$.

2 If $2\angle^s$ of one $\triangle$, are $= 2\angle^s$ of another $\triangle$, each to each, the third $\angle$ of the one, will be $=$ the third $\angle$ of the other.

3 In a Right-angled $\triangle$, the two Acute $\angle^s$ taken together, are preciſely $=$ a $r\angle$, *viz.* 90° Degrees.
Conſequently, if one of the Acute $\angle^s$ be given, the other is alſo given; *viz.* 90° Degrees — the given $\angle$ leaves the other $\angle$.

4 Each $\angle$ of an Equilateral $\triangle$ is $=$ 60° Degrees, or the third of $2\,r\angle^s$, which make 180° Degrees.

5 If one $\angle$ of any plain $\triangle$ be given, the Sum of the other $2\angle^s$ is alſo given, ſince 180° Degrees — the given $\angle$ is $=$ the other $2\angle^s$.

6 In an Iſoſceles $\triangle$, if the $\angle$ comprehended between the equal Legs be $=$ a $r\angle$, each of thoſe at the Baſe will be $= \frac{1}{2}\,r\angle$.

THE-

## THEOREM I.

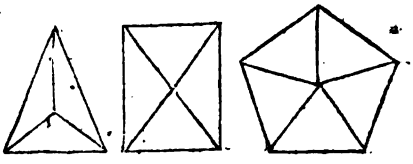
*All the Angles of a Right-lined Figure do together make twice as many Right-angles, bating four, as there are Sides of the Figure.*

## PREPARATION.

From any Point within the Figure, let Right-lines be drawn to all the Angles of the Figure, which ſhall reſolve the Figure into as many Triangles as there are Sides of the Figure.

I ſay, **All the ∠ˢ of the Figure are = twice as many r∠ˢ, bating four, as there are Sides.**

## DEMONSTRATION.



1 The ∠ˢ of every △ in the Figure are = 2 r∠ˢ. 32.
∵ 2 The ∠ˢ of all the △ˢ in the Figure are = twice as many r∠ˢ as there are Sides.
But 3 The ∠ˢ at the Point within the Figure are = 4r∠ˢ. *Cor.* 2. 15.
∴ 4 The remaining ∠ˢ are = twice as many r∠ˢ, bating four, as there are Sides.
i.e. 5 **All the ∠ˢ of the Figure are = twice as many r∠ˢ, bating four, as there are Sides.** Q. E. D.

## COROLLARIES.

1 Hence, all Right-lined Figures of the ſame Species have the Sums of their Angles equal.

2 In a Figure of four Sides, the four Angles make together four Right.

## THEOR. II.

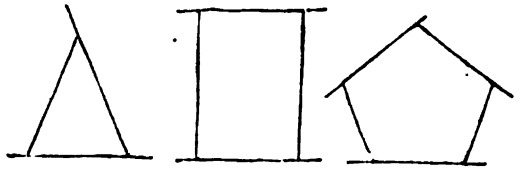
*All the outward Angles of any Right-lined Figure, taken together, make up four Right-angles.*

## PREPARATION.

Produce the Sides of the Figure.

I ſay, **The outward ∠ˢ are = 4 r∠ˢ.**

## DEMONSTRATION.



1 All the ſeveral Inward, with the ſeveral Outward ∠ˢ of the Figure are = 2.r∠ˢ. 13.

∵ 2 All the Inward, together with the Outward ∠ˢ of the Figure, are = twice as many r∠ˢ as there are Sides.

But 3 All the Inward ∠ˢ, with four right, are = twice as many r∠ˢ as there are Sides. *Theor.* 1.

∵ 4 **The outward ∠ˢ are = 4 r∠ˢ.** Q. E. D.

COROL-

## COROLLARY.

All Right-lined Figures of whatſoever Species have the Sums of their outward Angles equal.

## PROP. XXXIII.

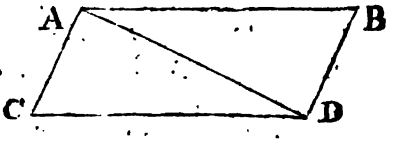
*If two Lines* AB, CD, *are parallel and equal, the Right-lines* AC, BD, *which join their Extremities, are alſo parallel and equal.*

## PREPARATION.

Draw the Diagonal AD.

I ſay, $\mathfrak{A}\mathfrak{C} \parallel$ and $= \mathfrak{B}\mathfrak{D}$.

## DEMONSTRATION.



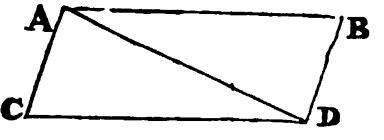

| | | |
|---|---|---|
| | 1 | AB = CD. *Hyp.* |
| And | 2 | AD is common. |
| And | 3 | ∠ADC = ∠DAB 29. |
| ∵ e. 1. 2. 3. | 4 | AB = CD, AD is common, ∠ADC = ∠DAB. |
| ∵ e. 4. | 5 | AC = BD. 4. |
| Alſo e. 4. | 6 | ∠DAC = ∠ADB. 4 |
| ∵ e. 6. | 7 | AC ∥ BD. 27. |
| ∵ e. 7. 5. | 8 | $\mathfrak{A}\mathfrak{C} \parallel$ and $= \mathfrak{B}\mathfrak{D}$. Q. E. D. |

## PROP. XXXIV.

*In Parallelograms, as* ABDC, *the oppoſite Sides* AB, CD, *and* AC, BD, *are equal each to other; and the oppoſite Angles,* C, B, *and* CAB, CDB, *are alſo equal; and the Diagonal* AD, *divides the ſame into two equal Parts or Triangles* ADB, ADC.

That is, $\mathfrak{AB} = \mathfrak{CD}$. $\mathfrak{AC} = \mathfrak{BD}$. $\angle\mathfrak{C} = \angle\mathfrak{B}$. $\angle\mathfrak{CAB} = \angle\mathfrak{CDB}$. $\triangle\mathfrak{ADB} = \triangle\mathfrak{ADC}$.

DEMONSTRATION.



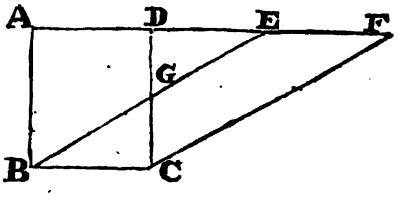

| | | |
|---|---|---|
| | 1 | AB ‖ CD. *Def.* 35. |
| ∵ e. 1. | 2 | ∠DAB = ∠ADC. 29. |
| Again | 3 | AC ‖ BD. *Def.* 35. |
| ∵ e. 3. | 4 | ∠DAC = ∠ADB. 29. |
| 2 + 4. | 5 | ∠DAB + ∠DAC = ∠ADC + ∠ADB. *Ax.* 11. |
| i.e | 6 | ∠CAB = ∠CDB. |
| Laſtly | 7 | AD is common. |
| ∵ e. 2. 4. 7. | 8 | ∠DAB = ∠ADC, ∠DAC = ∠ADB, AD is common. |
| ∵ e. 8. | 9 | AB = CD, AC = BD, ∠C = ∠B, △ADB = △ADC. 26. |
| ∴ 9. 6. | 10 | $\mathfrak{AB} = \mathfrak{CD}$. $\mathfrak{AC} = \mathfrak{BD}$. $\angle\mathfrak{C} = \angle\mathfrak{B}$. $\angle\mathfrak{CAB} = \angle\mathfrak{CDB}$. $\triangle\mathfrak{ADB} = \triangle\mathfrak{ADC}$. Q. E. D. |

PROP.

## PROP. XXXV.

*Parallelograms*, ABCD, BCFE, *which ſtand upon the ſame Baſe* BC, *and between the ſame Parallels* AF, BC, *are equal one to the other.*

That is, ▭ABCD = ▭BCFE.

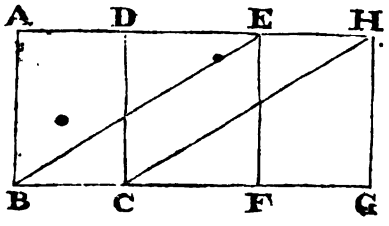



DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AD = BC. 34. |
| Alſo | 2 | EF = BC. 34. |
| ∵ e. 1. 2. | 3 | AD = EF. *Ax.* 1. |
| 3 + DE | 4 | AD + DE = DE + EF. *Ax.* 11. |
| i. e. | 5 | AE = DF. |
| Again | 6 | AB = DC. 34. |
| And | 7 | ∠A = ∠CDF. 29. |
| ∵ e. 5. 6. 7. | 8 | AE = DF, AB = DC, ∠A = ∠CDF. |
| ∵ e. 8. | 9 | △ABE = △DCF. 4. |
| 9 − △DGE | 10 | △ABE − △DGE = △DCF − △DGE. *Ax.* 12. |
| i. e. | 11 | Trapezium ABGD = Trapezium EGCF. |
| 11 + △BGC | 12 | Trap. ABGD + △BGC = Trap. EGCF + △BGC. *Ax.* 11. |
| i. e. | 13 | ▭ABCD = ▭BCFE. Q. E. D. |

## PROP. XXXVI.

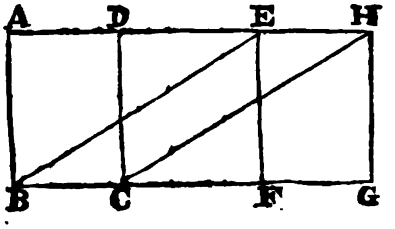
*Parallelograms*, ABCD, EFGH, *that are between the ſame Parallels* AH, BG, *and on equal Baſes* BC, FG, *are equal.*



## PREPARATION.

Join BE, CH.

I say, ▭ABCD = ▭EFGD.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | BC = FG. *Hyp.* |
| Alſo | 2 | EH = FG. 34. |
| ∵ e. 1. 2. | 3 | BC = EH. *Ax.* 1. |
| Alſo | 4 | BC ‖ EH. *Conſt.* |
| ∵ e. 4. 3. | 5 | BC ‖ and = EH. |
| ∵ e. 5. | 6 | BE ‖ CH. 33. |
| ∵ e. 5. 6. | 7 | BC ‖ EH, BE ‖ CH. |
| ∵ e. 7. | 8 | BCHE is a ▭. *Def.* 35. |
| But | 9 | ▭BCHE = ▭ABCD. 35. |
| Alſo | 10 | ▭BCHE = ▭EFGH. 35. |
| ∵ e. 9. 10. | 11 | ▭ABCD = ▭EFGD. *Ax.* 1. Q. E. D. |

## PROP. XXXVII.

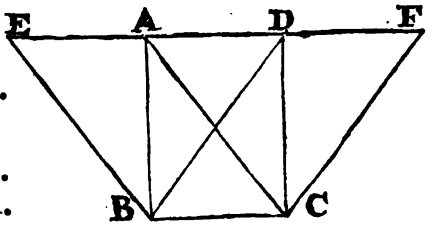
*Triangles*, BCA, BCD, *ſtanding upon the ſame Baſe* BC, *and between the ſame Parallels* BC, AD, *are equal one to the other.*

## PREPARATION.

Produce AD both ways, and through B draw (31.) BE ‖ CA, alſo through C draw (31.) CF ‖ BD, ſo (*Def.* 35.) are BCAE, BCFD, Parallelograms.

I ſay, △BCA = △BCD.

DEMON-




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\square BCFD = \square BCAE$. 35. |
| 1 ÷ 2 | 2 | $\frac{1}{2}\square BCFD = \frac{1}{2}\square BCAE$. *Ax.* 14. |
| Alſo | 3 | $\triangle BCA = \frac{1}{2}\square BCAE$. 34. |
| ∵ c. 3. 2. | 4 | $\triangle BCA = \frac{1}{2}\square BCFD$. *Ax.* 8. |
| Alſo | 5 | $\triangle BCD = \frac{1}{2}\square BCFD$. 34. |
| ∵ c. 4. 5. | 6 | $\triangle BCA = \triangle BCD$. *Ax.* 8. Q. E. D. |

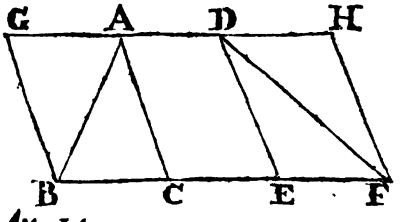
## PROP. XXXVIII.

*Triangles*, BCA, EFD, *ſet upon equal Baſes* BC, EF, *and between the ſame Parallels* AD, BF, *are equal the one to the other.*

## PREPARATION.

Produce AD both ways, and thro' B draw (31.) BG || CA, alſo thro' F draw (31.) FH || ED, ſo (*Def.* 35.) are BCAG, EFHD, Parallelograms.

I ſay, $\triangle BCA = \triangle EFD$.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\square EFHD = \square BCAG$. 36. |
| 1 ÷ 2 | 2 | $\frac{1}{2}\square EFHD = \frac{1}{2}\square BCAG$. *Ax.* 14. |
| Alſo | 3 | $\triangle BCA = \frac{1}{2}\square BCAG$. 34. |
| ∵ c. 3. 2. | 4 | $\triangle BCA = \frac{1}{2}\square EFHD$. *Ax.* 8 |
| Alſo | 5 | $\triangle EFD = \frac{1}{2}\square EFHD$. 34. |
| ∵ c. 4. 5. | 6 | $\triangle BCA = \triangle EFD$. *Ax.* 8. Q. E. D. |

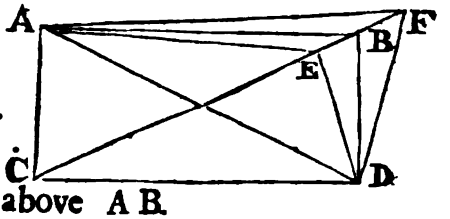
## PROP. XXXIX.

*Triangles*, ADC, BDC, *which have the ſame Baſe* CD, *and are equal to each other, are between the ſame Parallels* AB, CD.

## PREPARATION.

If the Parallel be ſaid to fall above AB, through A draw (31.) AF || CD, if below, through A draw (31.) AE || CD, and producing CB, 'till it meet AF, join FD, ED.

I ſay, AB || CD.




DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If the Parallel ſhould fall above AB. |
| Then | 2 | △ADC would be = △FDC. 37. |
| Alſo | 3 | △ADC = △BDC. *Hyp.* |
| ∵ e. 2. 3. | 4 | △FDC would be = △BDC. *Ax.* 1. |
| But | 5 | △FDC ⊏ △BDC. *Ax.* 17. |
| ∴ | 6 | The Parallel will not fall above AB. |
| Again | 7 | If the Parallel ſhould fall below AB. |
| Then | 8 | △ADC would be = △EDC. 37. |
| Alſo | 9 | △ADC = △BDC. *Hyp.* |
| ∵ e. 9. 8. | 10 | △BCD would be = △EDC. *Ax.* 1. |
| But | 11 | △BDC ⊏ △EDC. *Ax.* 17. |
| ∴ | 12 | The Parallel will not fall below AB. |
| ∵ e. 6. 12. | 13 | The Parallel falls neither above nor below AB. |
| ∴ e. 13. | 14 | AB || CD. Q. E. D. |

PROP.

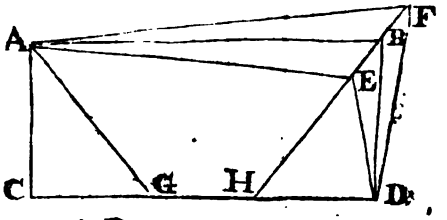
## PROP. XL.

*Equal Triangles*, AGC, BHD, *ſtanding upon equal Baſes* CG, DH; *and on the ſame Side, are between the ſame Parallels* AB, CD.

## PREPARATION.

If the Parallel be ſaid to fall above AB, draw (31.) AF ‖ CD, if below, draw (31.) AE ‖ CD, and producing HE till it meet AF, joyn FD, ED.

I ſay, 𝔄𝔅 ‖ ℭ𝔇.

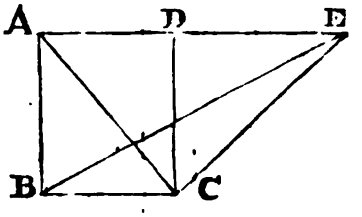



## DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If the Parallel ſhould fall above AB. |
| Then | 2 | △ AGC would be = △ FHD. 38. |
| Alſo | 3 | △ AGC = △ BHD. *Hyp.* |
| ∵ e. 2. 3. | 4 | △ FHD would be = △ BHD. *Ax.* 1. |
| But | 5 | △ FHD ⊏ △ BHD. *Ax.* 17. |
| ∴ | 6 | The Parallel will not fall above AB. |
| Again | 7 | If the Parallel ſhould fall below AB. |
| Then | 8 | △ AGC would be = △ EHD. 38. |
| Alſo | 9 | △ AGC = △ BHD. *Hyp.* |
| ∵ e. 9. 8. | 10 | △ BHD would be = △ EHD. *Ax.* 1. |
| But | 11 | △ BHD ⊏ △ EHD. *Ax.* 17. |
| ∴ | 12 | The Parallel will not fall below AB. |
| ∴ e. 6. 12. | 13 | The Parallel falls neither above nor below AB. |
| ∵ e. 13. | 14 | 𝔄𝔅 ‖ ℭ𝔇. Q. E. D. |

## PROP. XLI.

*If a Parallelogram* ABCD, *have the ſame Baſe* BC, *with the Triangle* BCE, *and be between the ſame Parallels* AE, BC, *then is the Parallelogram* ABCD, *double to the Triangle* BCE.




### PREPARATION.

Draw the Diagonal AC.

I ſay, $\square ABCD = 2\triangle BCE$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\triangle BCA = \triangle BCE$. 37. |
| Alſo | 2 | $\triangle BCA = \triangle CDA$. 34. |
| ∵ e. 1. 2. | 3 | $\triangle BCE = \triangle CDA$. *Ax.* 1. |
| $3 \times \frac{2}{}$ | 4 | $2\triangle BCE = 2\triangle CDA$. *Ax.* 13. |
| Alſo | 5 | $\square ABCD = 2\triangle CDA$. 34. |
| ∵ 5. 4. | 6 | $\square ABCD = 2\triangle BCE$. *Ax.* 7. Q. E. D. |

## PROP. XLII.

*To make a Parallelogram* EFGC, *equal to a given Triangle* ABC, *and having one Angle* GCE, *equal to a given right-lined Angle* D.

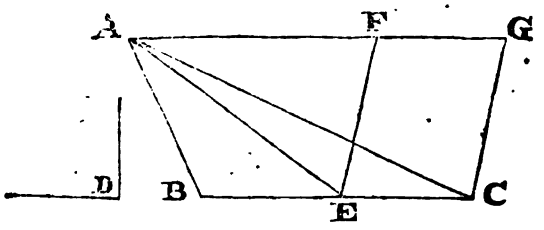
### PREPARATION.

Through A draw (31.) AG ‖ BC; then making (23.) ∠ GCE = ∠ D, and (10.) biſecting the Baſe BC in E, draw AE, and (31.) EF ‖ CG.

I ſay,

I say, ▭ EFGC = △ ABC. ∠ GCE = ∠ D.

DEMONSTRATION.

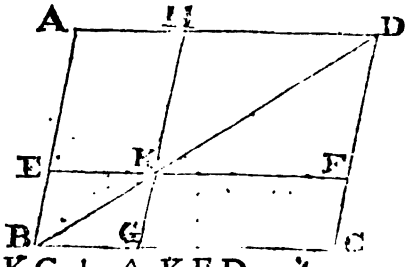



| | | |
|---|---|---|
| | 1 | BE = EC, AG ‖ BC. *Const.* |
| ∵ e. 1. | 2 | △ ABE = △ AEC. 38. |
| Again | 3 | △ ABC = △ ABE + △ AEC. *Ax.* 18. |
| ∵ e. 2. 3. | 4 | △ ABC = △ AEC + △ AEC. *Sub.* |
| i. e. | 5 | △ ABC = 2 △ AEC. |
| Alſo | 6 | ▭ EFGC = 2 △ AEC. 41. |
| ∵ e. 6. 5. | 7 | ▭ EFGC = △ ABC. *Ax.* 7. |
| Laſtly | 8 | ∠ GCE = ∠ D. *Const.* |
| ∵ e. 7. 8. | 9 | ▭ EFGC = △ ABC. ∠ GCE = ∠ D. Q. E. F. & D. |

PROP. XLIII.

*In every Parallelogram* ABCD, *the Complements* AK, KC, *of thoſe Parallelograms* EG, HF, *which are about the Diagonal* BD, *are equal.*

That is, ▭ AK = ▭ KC.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | △ EKB = △ BKG. 34. |
| Alſo | 2 | △ KHD = △ KFD. 34. |
| 1 + 2 | 3 | △ EKB + △ KHD = △ BKG + △ KFD. *Ax.* 11. |
| Again | 4 | △ ABD = △ BDC. 34. |
| i. e. | 5 | △ EKB + △ KHD + ▭ AK = △ BKG + △ KFD + ▭ KC. |
| 5 − 3 | 6 | ▭ AK = ▭ KC. *Ax.* 12. Q. E. D. |

## PROP. XLIV.

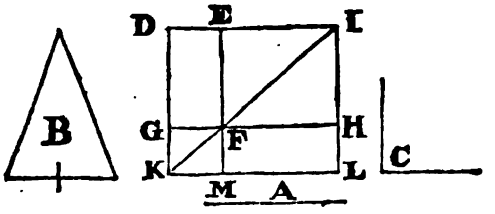
*Upon a Right-line given* A, *to make a Parallelogram* FL, *equal to a Triangle given* B, *and having an Angle* MFH, *equal to an Angle given* C.

## PREPARATION.

Make (42.) a $\square$FD $=$ $\triangle$ B, so that $\angle$ GFE be $=$ $\angle$ C. Produce GF till FH be $=$ A. Through H draw (31.) IL $\parallel$ EF, which let DE produced meet in I. Let DG drawn forth, meet with a Right-line drawn from I, in the Point K. Through K draw (31.) KL $\parallel$ GH, with which let EF drawn out meet at M, and IH at L.

I say, $\mathfrak{F}\mathfrak{H} = \mathfrak{A}$. $\square\mathfrak{F}\mathfrak{L} = \triangle\mathfrak{B}$. $\angle\mathfrak{M}\mathfrak{F}\mathfrak{H} = \angle\mathfrak{C}$.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | FH $=$ A. *Conſt.* |
| And | 2 | $\square$FL $=$ $\square$FD. 43. |
| Alſo | 3 | $\triangle$ B $=$ $\square$FD. *Conſt.* |
| ∵ e. 2. 3. | 4 | $\square$FL $=$ $\triangle$ B. *Ax.* 1. |
| Again | 5 | $\angle$ MFH $=$ $\angle$ GFE. 15. |
| Alſo | 6 | $\angle$ C $=$ $\angle$ GFE. *Conſt.* |
| ∵ e. 5. 6. | 7 | $\angle$ MFH $=$ $\angle$ C. *Ax.* 1. |
| ∵ e. 1. 4. 7. | 8 | $\mathfrak{F}\mathfrak{H} = \mathfrak{A}$. $\square\mathfrak{F}\mathfrak{L} = \triangle\mathfrak{B}$. $\angle\mathfrak{M}\mathfrak{F}\mathfrak{H} = \angle\mathfrak{C}$. Q. E. F. & D. |

PROP.

## PROP. XLV.

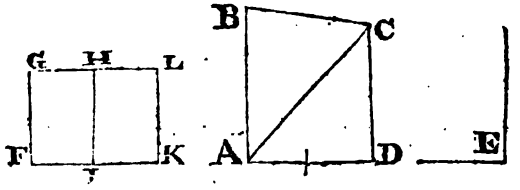
*To make a Parallelogram* GFKL, *equal to a given right-lined Figure* ABCD, *and having an Angle* F, *equal to a given Angle* E.

### PREPARATION.

Reſolve the right-lined Figure into Triangles ABC, ADC, and making (42.) □GFIH = △ABC, having ∠F = ∠E, produce the Line FI, and (42.) make □HIKL = △ADC.

I ſay, □GFKL = ABCD. ∠F = ∠E.

### DEMONSTRATION.



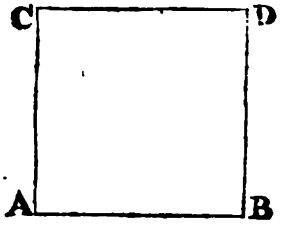

| | | |
|---|---|---|
| | 1 | □GFIH = △ABC. *Conſt.* |
| And | 2 | □HIKL = △ADC. *Conſt.* |
| 1 + 2 | 3 | □GFIH + □HIKL = △ABC + △ADC. *Ax.* 11. |
| i. e. | 4 | □GFKL = ABCD. |
| Again | 5 | ∠F = ∠E. *Conſt.* |
| ∴ e. 4. 5. | 6 | □GFKL = ABCD. ∠F = ∠E. Q. E. F. & D. |

## PROP. XLVI.

*Upon a Line given* A B, *to deſcribe a Square* A B D C.

## PREPARATION.

Erect (11.) two Perpendiculars A C, B D, equal to A B, and join C D.




I ſay, A B D C is a Square.

## DEMONSTRATION.

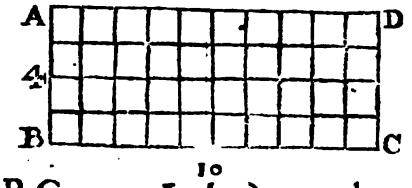
| | | |
|---|---|---|
| | 1 | A, B, are r∠s. *Def.* 10. |
| ∵ e. 1. | 2 | A C ‖ B D. 28. |
| Again | 3 | A C = A B = B D. *Conſt.* |
| ∵ e. 2. 3. | 4 | A C ‖ and = B D. |
| ∵ e. 4. | 5 | A B ‖ and = C D. 33. |
| ∵ e. 4. 5. | 6 | A C ‖ B D, A B ‖ C D. |
| ∵ e. 6. | 7 | A B D C is a □. *Def.* 35. |
| ∵ e. 7. | 8 | ∠ A = ∠ D, ∠ B = ∠ C. 34. |
| But e. 1. | 9 | A, B, are r∠s. |
| ∵ e. 8. 9. | 10 | D, C, are r∠s. |
| ∵ e. 9. 10. | 11 | ∠ A = ∠ B = ∠ D = ∠ C. *Ax.* 21. |
| Alſo e. 3. 5. | 12 | A C = A B = B D = C D. *Ax.* 1. |
| ∵ e. 12. 11. | 13 | A C = A B = B D = C D, ∠ A = ∠ B = ∠ D = ∠ C. |
| ∵ e. 13. | 14 | A B D C is a Square. *Def.* 29. Q. E. F. & D. |

LEM-

## LEMMA I.

The Area of every Square, Right Angled Parallelogram, or Rectangle, is that Space which is contain'd within its Sides, being equal to the Product of those Numbers, which express the Measure of its Sides.

Thus the Area of the Right-angled Parallelogram, or Rectangle A B C D, is that Space which is contained within its four Sides, A B, B C, C D, D A, being (if A B be = 4, B C = 10 Inches) equal to A B × B C, = 4 × 10 = 40 Square Inches.




## COROLLARY.

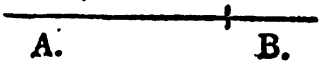
Every Right-angled Parallelogram therefore, or Rectangle A B C D, is said to be contain'd under two Right-lines A B, B C, comprehending a Right-angle B.

Wherefore, by A B × B C, or such like, is always meant the Rectangle contain'd under the Right-lines, A B, B C, set at Right-angles; by Z A = Z × A, or such like, is always meant the Rectangle contained under the Right-lines Z, A, set at Right-angles.

## LEMMA II.

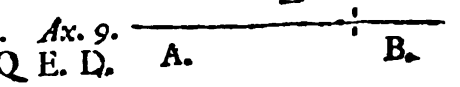
If a Right-line Z, be cut any-wise into two Parts, the Square made of the whole Z, is equal both to the Squares made of the Segments A, B, and twice a Rectangle made of the Parts A, B.

That is, $\mathfrak{Z}\mathfrak{q} = \mathfrak{A}\mathfrak{q} + \mathfrak{B}\mathfrak{q} + 2\,\mathfrak{A}\mathfrak{B}$.



## DEMONSTRATION.

1 $Z = A + B$. *Ax.* 18.
2 2 $ZZ = AA + BB + 2AB$. *Ax.* 9.
i. e. 3 $\mathfrak{Z}\mathfrak{q} = \mathfrak{A}\mathfrak{q} + \mathfrak{B}\mathfrak{q} + 2\mathfrak{A}\mathfrak{B}$. Q. E. D.



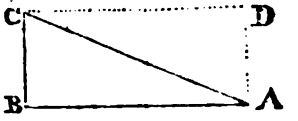

## LEMMA III.

The Area of every Right-ang'ed Triangle A B C, is equal to half the Parallelogram made of its Base A B, and Perpendicular B C.

## PREPARATION.

Compleat the Right-angled Parallelogram A B C D.

I say, $\triangle \mathfrak{A}\mathfrak{B}\mathfrak{C} = \frac{1}{2} \square \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$.




## DEMONSTRATION.

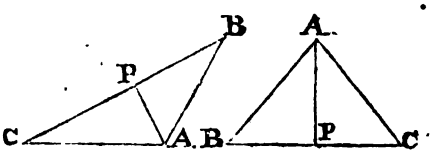
1 $\triangle \mathfrak{A}\mathfrak{B}\mathfrak{C} = \frac{1}{2} \square \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$. 34. Q. E. D.

## COROLLARIES.

1 Because the Area of the $\square$ A B C D is (*Lem.* 1.) equal to its Base A B, drawn into its Perpendicular B C, and the $\triangle$ A B C is equal to $\frac{1}{2} \square$ A B C D; therefore the Area of a Right-angled Triangle A B C, is equal to its Base A B, drawn into half its Perpendicular B C.

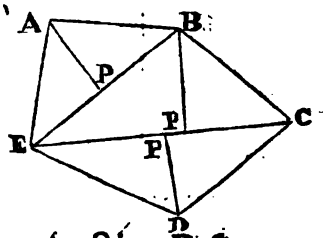
Hence

2 Hence the Areas of Acute and Obtuſe-angled Triangles A B C, may be alſo found, *viz.* by reſolving them into Right-angled Triangles,




by Means of a Perpendicular A P, let fall from the Acute and Obtuſe Angles upon the oppoſite Side. Thus the Areas of the Acute and Obtuſe-angled Triangles, A B C, are $= \frac{1}{2}$ A P $\times$ P B $+ \frac{1}{2}$ A P $\times$ P C $= \frac{1}{2}$ A P $\times$ B C.

3 Hence alſo the Area of any irregular Figure, or Field A B C D E, may be found, *viz.* by reſolving it into Triangles; and each of thoſe Triangles into Right-angled Ones, by means of Perpendiculars let fall from their vertical Angles upon their reſpective Baſes.

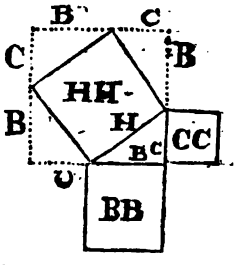



Thus the Area of the irregular Figure or Field A B C D E, is $= \frac{1}{2}$ A P $\times$ B E $+ \frac{1}{2}$ B P $\times$ C E $+ \frac{1}{2}$ D P $\times$ C E.

## PROP. XLVII.

*In any Right-angled Triangle* B C H, *the Square* H H, *deſcrib'd upon the Hypothenuſe* H, *is equal to the Sum of the Squares,* B B, C C, *deſcribed on the other two Sides* B, C.

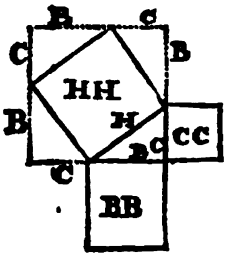
### PREPARATION.




Make (46.) a Square whoſe Side is $= B + C$, and let that Square be expreſs'd by S, ſo (*Lem.* 2.) will its Area be $= B B + C C + 2 B C$, alſo the Sum of the Areas of the Four Triangles included in that Square, will (*Cor.* 1. *Lem.* 3.) be $= \frac{1}{2} B C \times 4 = 2 B C$.

I say, $\mathfrak{H}\mathfrak{H} = \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C}$.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $S = HH + 2BC$. |
| Alſo | 2 | $S = BB + CC + 2BC$. |
| ∴ e. 1. 2. | 3 | $HH + 2BC = BB + CC + 2BC$. *Ax.* 1. |
| 3 — 2BC. | 4 | $\mathfrak{H}\mathfrak{H} = \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C}$. *Ax.* 12. Q. E. D. |

## COROLLARIES.

| | | |
|---|---|---|
| | 1 | $\mathfrak{H} = \sqrt{\mathfrak{C}q + \mathfrak{B}q}$. |
| For | 1 | $HH = CC + BB$. |
| i. e. | 2 | $Hq = Cq + Bq$. |
| 2 in 2 | 3 | $\mathfrak{H} = \sqrt{\mathfrak{C}q + \mathfrak{B}q}$. *Ax.* 10. |
| | 2 | $\mathfrak{B} = \sqrt{\mathfrak{H}q - \mathfrak{C}q}$. |
| For | 1 | $Bq + Cq = Hq$. |
| 1 — Cq. | 2 | $Bq = Hq - Cq$. *Ax.* 12. |
| 2 in 2. | 3 | $\mathfrak{B} = \sqrt{\mathfrak{H}q - \mathfrak{C}q}$. *Ax.* 10. |
| | 3 | $\mathfrak{C} = \sqrt{\mathfrak{H}q - \mathfrak{B}q}$. |
| For | 1 | $Cq + Bq = Hq$. |
| 1 — Bq | 2 | $Cq = Hq - Bq$. *Ax.* 12. |
| 2 in 2 | 3 | $\mathfrak{C} = \sqrt{\mathfrak{H}q - \mathfrak{B}q}$. *Ax.* 10. |

## PROP. XLVIII.

*If a Square made upon one Side* BC, *of a Triangle* ABC, *be equal to the Squares made on the other Sides* AB, AC, *the Angle* CAB, *compreheneaa under the other Sides* AB, AC, *is Right.*

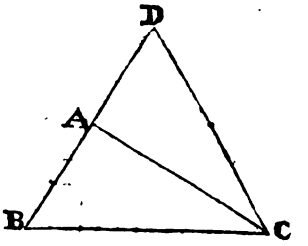
PRE-

## PREPARATION.

Draw to the Point A in AC, the Perpendicular DA = AB, and joyn CD.

I say, CAB is a r∠.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | CAD is a r∠. *Def.* 10. |
| ∴ e. 1. | 2 | $CD^q = AC^q + DA^q$. 47. |
| Again | 3 | DA = AB. *Const.* |
| 3 & 2. | 4 | $DA^q = AB^q$. *Ax.* 9. |
| ∴ e. 2. 4. | 5 | $CD^q = AC^q + AB^q$. *Sub.* |
| Also | 6 | $BC^q = AC^q + AB^q$. *Hyp.* |
| ∴ e. 5. 6. | 7 | $CD^q = BC^q$. *Ax.* 1. |
| 7 & 2 | 8 | CD = BC. *Ax.* 10. |
| Lastly | 9 | AC is common. |
| ∴ e. 3. 8. 9. | 10 | DA = AB, CD = BC, AC is common. |
| ∴ e. 10. | 11 | ∠ CAD = ∠ CAB. 8. |
| But e. 1. | 12 | CAD is a r∠. |
| ∴ e. 11. 12. | 13 | CAB is a r∠. Q. E. D. |

*The End of the First Book.*

# EUCLID's ELEMENTS,

Demonſtrated after a New, Plain, and Eaſie METHOD.

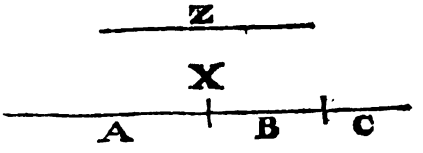
## BOOK II.

### PROPOSITION I.

*If two Right-lines* Z, X, *be given, and one of them* X, *divided into as many Parts or Segments as you pleaſe; the Rectangle contained under the two whole Right-lines* Z, X, *ſhall be equal to all the Rectangles contained under the whole Line* Z, *and the ſeveral Segments* A, B, C.

That is, $\mathfrak{Z}\mathfrak{X} = \mathfrak{Z}\mathfrak{A} + \mathfrak{Z}\mathfrak{B} + \mathfrak{Z}\mathfrak{C}$.

### DEMONSTRATION.



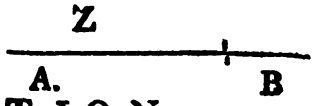

1 | $X = A + B + C$. *Ax.* 18.
1 x Z 2 | $\mathfrak{Z}\mathfrak{X} = \mathfrak{Z}\mathfrak{A} + \mathfrak{Z}\mathfrak{B} + \mathfrak{Z}\mathfrak{C}$. *Ax.* 13. Q. E. D.

PROP.

## PROP. II.

*If a Right-line* Z, *be divided anywiſe into two Parts, the Square made of the whole Line* Z, *is equal to the Rectangles contained under the whole Line* Z, *and each of the Segments* A, B.

That is, $\mathfrak{Z}\mathfrak{q} = \mathfrak{Z}\mathfrak{A} + \mathfrak{Z}\mathfrak{B}$.



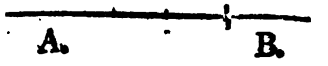

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $Z = A + B$. *Ax.* 18. |
| 1 × Z | 2 | $ZZ = ZA + ZB$. *Ax.* 13. |
| i. e. | 3 | $\mathfrak{Z}\mathfrak{q} = \mathfrak{Z}\mathfrak{A} + \mathfrak{Z}\mathfrak{B}$. Q. E. D. |

## PROP. III.

*If a Right-line* Z, *be divided anywiſe into two Parts, the Rectangle contained under the whole Line* Z, *and one of the Segments* B, *is equal to the Rectangle made of the Segments* A, B, *and the Square deſcribed on the ſaid Segment* B.

That is, $\mathfrak{Z}\mathfrak{B} = \mathfrak{A}\mathfrak{B} + \mathfrak{B}\mathfrak{q}$.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $Z = A + B$. *Ax.* 18. |
| 1 × B | 2 | $ZB = AB + BB$. *Ax.* 13. |
| i. e. | 3 | $\mathfrak{Z}\mathfrak{B} = \mathfrak{A}\mathfrak{B} + \mathfrak{B}\mathfrak{q}$. Q. E. D. |

## PROP. IV.

*If a Right-line* Z, *be cut anywise into two Parts, the Square made of the whole* Z, *is equal both to the Squares made of the Segments* A, B, *and twice a Rectangle made of the Parts* A, B.

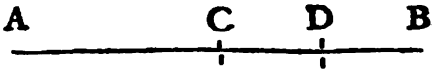
That is, $\mathfrak{Z}q = \mathfrak{A}q + \mathfrak{B}q + 2\,\mathfrak{A}\mathfrak{B}$.

### DEMONSTRATION.

1 $\mathfrak{Z}q = \mathfrak{A}q + \mathfrak{B}q + 2\,\mathfrak{A}\mathfrak{B}$. *Lem.* 2. 1. Q. E. D.

## PROP. V.

*If a Right-line* AB, *be cut equally in* C, *and unequally in* D, *the Rectangle contained under the unequal Parts* AD, DB, *together with the Square that is made of the Difference of the Parts* CD, *is equal to the Square that is made of the Half-line* CB.




That is, $\mathfrak{A}\mathfrak{D} \times \mathfrak{D}\mathfrak{B} + \mathfrak{C}\mathfrak{D}q = \mathfrak{C}\mathfrak{B}q$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $CDq + DBq + \overline{CD \times DB} \times 2 = CBq$. 4. |
| And | 2 | $CB \times DB = CD \times DB + DBq$. 3. |
| ∴ e. 1. 2. | 3 | $CDq + CB \times DB + CD \times DB = CBq$. *Sub.* |
| Again | 4 | $AC = CB$. *Hyp.* |
| ∴ e. 3. 4. | 5 | $CDq + AC \times DB + CD \times DB = CBq$. *Sub.* |
| Lastly | 6 | $AD \times DB = AC \times DB + CD \times DB$. 1. |
| ∴ e. 5. 6. | 7 | $\mathfrak{A}\mathfrak{D} \times \mathfrak{D}\mathfrak{B} + \mathfrak{C}\mathfrak{D}q = \mathfrak{C}\mathfrak{B}q$. *Sub.* Q. E. D. |

PROP.

## PROP. VI.

*If a Right-line* A, *be divided into two equal Parts, and another Right-line* B, *added to the same directly in one Right-line; then the Rectangle contained under the whole and the Line added, viz.* A + B, *and the Line added* B, *together with the Square which is made of one half the Line* A, *is equal to the Square of* ½ A + B, *taken as one Line.*

That is, $\overline{\mathfrak{A}+\mathfrak{B}}\times\mathfrak{B}+\frac{1}{4}\mathfrak{A}\mathfrak{A}=\overline{\frac{1}{2}\mathfrak{A}+\mathfrak{B}}.^2$

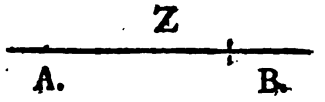
DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $AB+BB=\overline{A+B}\times B.$ |
| 1 + ¼ AA | 2 | $AB+BB+\frac{1}{4}AA=\overline{A+B}\times B+\frac{1}{4}AA.$ *Ax.* 11. |
| Alſo | 3 | $AB+BB+\frac{1}{4}AA=\overline{\frac{1}{2}A+B}.^2$ |
| ∵ c. 2. 3. | 4 | $\overline{\mathfrak{A}+\mathfrak{B}}\times\mathfrak{B}+\frac{1}{4}\mathfrak{A}\mathfrak{A}=\overline{\frac{1}{2}\mathfrak{A}+\mathfrak{B}}.^2$ *Ax.* 1. Q. E. D. |

## PROP. VII.

*If a Right-line* Z, *be divided anywiſe into two Parts, the Square made of the whole Line* Z, *together with the Square made of one of the Segments* B, *is equal to a double Rectangle contained under the whole Line* Z, *and the ſaid Segment* B, *together with the Square made of the other Segment* A.

That is, $\mathfrak{Z}\mathfrak{q}+\mathfrak{B}\mathfrak{q}=2\mathfrak{Z}\mathfrak{B}+\mathfrak{A}\mathfrak{q}.$




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $ZB=AB+Bq.$ 3. |
| 1 × 2 | 2 | $2ZB=2AB+2Bq.$ *Ax.* 13. |
| 2 − 2 Bq | 3 | $2ZB-2Bq=2AB.$ *Ax.* 12. |
| Again | 4 | $Zq=Aq+Bq+2AB.$ 4. |
| ∵ c. 3. 4. | 5 | $Zq=Aq+Bq+2ZB-2Bq.$ *Sub.* |
| i. c. | 6 | $Zq=Aq+2ZB-Bq.$ |
| 6 + Bq. | 7 | $\mathfrak{Z}\mathfrak{q}+\mathfrak{B}\mathfrak{q}=2\mathfrak{Z}\mathfrak{B}+\mathfrak{A}\mathfrak{q}.$ Q. E. D. |

## PROP. VIII.

*If a Right-line* Z, *be divided anywise into two Parts, the Rectangle contained under the whole Line* Z, *and one of the Segments* B *four Times, together with the Square of the other Segment* A, *is equal to the Square of the whole Line* Z, *and the other Segment* B, *viz. of* Z + B, *taken as one Line.*

That is, $4\,Z\,B + A\,q = \overline{Z + B}.^2$

Z

A. B

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $Z\,q + B\,q = 2\,Z\,B + A\,q$. 7. |
| 1 + 2 ZB | 2 | $Z\,q + B\,q + 2\,Z\,B = 4\,Z\,B + A\,q$. *Ax.* 11. |
| Alſo | 3 | $Z\,q + B\,q + 2\,Z\,B = \overline{Z + B}.^2$ |
| ∴ 6. 2. 3. | 4 | $4\,Z\,B + A\,q = \overline{Z + B}.^2$ *Ax.* 1. Q. E. D. |

## PROP. IX.

*If a Right-line* A B, *be cut equally in* C, *and unequally in* D, *the Squares of the unequal Parts* A D, D B, *taken together, are double to the Squares of the Half-line* A C, *and the Difference of the Parts* C D.

That is, $A\,D\,q + D\,B\,q = 2\,A\,C\,q + 2\,C\,D\,q$.

DEMONSTRATION.

A C D B

A D q

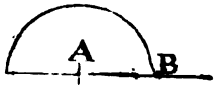
| | | |
|---|---|---|
| | 1 | $AD\,q = AC\,q + CD\,q + \overline{AC \times CD} \times 2$. 4. |
| 1 + DB q | 2 | $AD\,q + DB\,q = AC\,q + CD\,q + \overline{AC \times CD} \times 2 + DB\,q$. *Ax.* 11. |
| Again | 3 | $AC = CB$. *Hyp.* |
| ∴ c. 2. 3. | 4 | $AD\,q + DB\,q = AC\,q + CD\,q + \overline{CB \times CD} \times 2 + DB\,q$. *Sub.* |
| Laſtly | 5 | $\overline{CB \times CD} \times 2 + DB\,q = CB\,q + CD\,q$. 7. |
| 3 & 2 | 6 | $AC\,q = CB\,q$. *Ax.* 9. |
| ∴ c 5. 6. | 7 | $\overline{CB \times CD} \times 2 + DB\,q = AC\,q + CD\,q$. *Sub.* |
| ∴ c. 4. 7. | 8 | $AD\,q + DB\,q = AC\,q + CD\,q + AC\,q + CD\,q$. *Sub.* |
| i. e. | 9 | $\mathfrak{A}\mathfrak{D}\,q + \mathfrak{D}\mathfrak{B}\,q = 2\,\mathfrak{A}\mathfrak{C}\,q + 2\,\mathfrak{C}\mathfrak{D}\,q$. Q. E. D. |

## PROP. X.

*If a Right-line* A, *be divided into two equal Parts, and another Line be added in a Right-line with the ſame; then is the Square of the whole Line, together with the added Line, as being one Line, together with the Square of the added Line* B, *double to the Square of half* A, *and the added Line* B, *taken as one Line, and to the Square of* ½ A.

That is, $\overline{\mathfrak{A} + \mathfrak{B}}^2 + \mathfrak{B}\,q = \overline{\frac{1}{2}\mathfrak{A} + \mathfrak{B}}^2 \times 2 + \frac{1}{2}\mathfrak{A}\mathfrak{A}$.

### DEMONSTRATION.



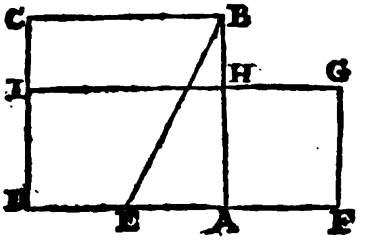

| | | |
|---|---|---|
| | 1 | $\overline{A + B}^2 = A\,q + B\,q + 2\,AB$. |
| 1 + B q. | 2 | $\overline{A + B}^2 + B\,q = A\,q + 2\,B\,q + 2\,AB$. *Ax.* 11. |
| Again | 3 | $\overline{\frac{1}{2}A + B}^2 = \frac{1}{4}AA + B\,q + AB$. |
| 3 × 2 | 4 | $\overline{\frac{1}{2}A + B}^2 \times 2 = \frac{1}{2}AA + 2\,B\,q + 2\,AB$. *Ax.* 13. |
| 4 + ½ AA | 5 | $\overline{\frac{1}{2}A + B}^2 \times 2 + \frac{1}{2}AA = A\,q + 2\,B\,q + 2\,AB$. *Ax.* 11. |
| ∴ c. 2. 5. | 6 | $\overline{\mathfrak{A} + \mathfrak{B}}^2 + \mathfrak{B}\,q = \overline{\frac{1}{2}\mathfrak{A} + \mathfrak{B}}^2 \times 2 + \frac{1}{2}AA$. *Ax.* 1. Q. E. D. |

## PROP. XI.

*To cut a Right-line* A B, *in the Point* H, *ſo that the Square of one of its Parts* A H, *may be equal to the Rectangle contained under the whole Line* A B, *and the other Part* B H.

### PREPARATION.

Upon A B deſcribe (46. 1.) the Square A B C D. Biſect (10. 1.) D A in E, and draw E B. From the Line D A produced take E F = E B. On A F make (46. 1.) the Square A G, and extend G H to L.




I ſay, $\square AH = \square AB \times BH$.

DEMONSTRATION.

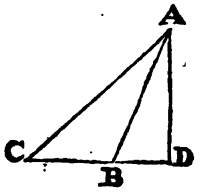
| | | |
|---|---|---|
| | 1 | $DF \times AF + EA\,q = EF\,q$. 6. |
| And | 2 | $AF = FG$. *Def.* 29. 1. |
| ∴ e. 1. 2. | 3 | $DF \times FG + EA\,q = EF\,q$. *Sub.* |
| i. e. | 4 | $DG + EA\,q = EF\,q$. |
| Again | 5 | $EB = EF$. *Conſt.* |
| 5 ⊙ 2. | 6 | $EB\,q = EF\,q$. *Ax.* 9. |
| ∴ e. 4. 6. | 7 | $DG + EA\,q = EB\,q$. *Sub.* |
| Alſo | 8 | $EA\,q + AB\,q = EB\,q$. 47. 1. |
| ∴ e. 7. 8. | 9 | $DG + EA\,q = EA\,q + AB\,q$. *Ax.* 1. |
| 9 — $EA\,q$ | 10 | $DG = AB\,q$. *Ax.* 12. |
| 10 — $AL$. | 11 | $DG - AI = AB\,q - AL$. *Ax.* 12. |
| i. e. | 12 | $AH\,q = BL$ |
| Alſo | 13 | $BC \times BH = BL$ |
| ∴ e. 12. 13. | 14 | $AH\,q = BC \times BH$. *Ax.* 1. |
| Laſtly | 15 | $AB = BC$. *Def.* 29. 1. |
| ∴ e. 14. 15. | 16 | $\square AH = \square AB \times BH$. *Sub.* Q. E. D. |

PROP.

## PROP. XII.

*In Obtuse-angled Triangles* ABC, *the Square that is made of the Side* AC, *subtending the Obtuse Angle* ABC, *is greater than the Squares of the Sides* AB, CB, *that contain the Obtuse-Angle* ABC, *by a double Rectangle contained under one of the Sides* CB, *which are about the Obtuse-Angle* ABC, *on which Side produced the Perpendicular* AD *falls, and under the Line* BD, *taken without the Triangle from the Point on which the Perpendicular* AD *falls, to the Obtuse-Angle* ABC.

That is, $\mathfrak{AC}q = \mathfrak{AB}q + \mathfrak{CB}q + \overline{\mathfrak{CB} \times \mathfrak{BD}} \times 2$.

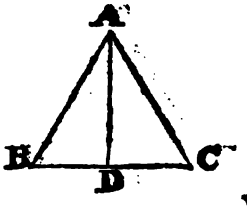



### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $AC q = AD q + CD q$. 47. 1. |
| And | 2 | $CD q = BD q + CB q + \overline{CB \times BD} \times 2$. 4. |
| ∵ e. 1. 2. | 3 | $AC q = AD q + BD q + CB q + \overline{CB \times BD} \times 2$. *Sub.* |
| Again | 4 | $AB q = AD q + BD q$. 47. 1. |
| ∵ e. 3. 4. | 5 | $\mathfrak{AC}q = \mathfrak{AB}q + \mathfrak{CB}q + \overline{\mathfrak{CB} \times \mathfrak{BD}} \times 2$. *Sub.* Q. E. D. |

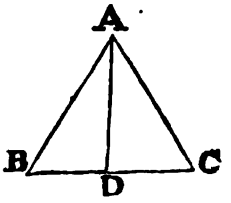
## PROP. XIII.

*In Acute-angled Triangles* ABC, *the Square made of the Side* AB, *subtending the Acute-angle* ACB, *is less than the Squares made on the Sides* AC, BC, *comprehending the Acute-angle* ACB, *by a double Rectangle contained under one of the Sides* BC, *which are about the Acute-angle* ACB, *on which the Perpendicular* AD *falls, and under the Line* DC, *taken within the Triangle, from the Perpendicular* AD, *to the Acute-angle* ACB.



That is, $\mathfrak{A}\mathfrak{B}q + \overline{\mathfrak{B}\mathfrak{C} \times \mathfrak{D}\mathfrak{C}} \times 2 = \mathfrak{A}\mathfrak{C}q + \mathfrak{B}\mathfrak{C}q$.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $AD\,q + DC\,q = AC\,q$. 47. 1. |
| $\ldots + BC\,q$ | 2 | $AD\,q + BC\,q + DC\,q - AC\,q + BC\,q$. *Ax.* 11. |
| Again | 3 | $BC\,q + DC\,q = \overline{BC \times DC} \times 2 + BD\,q$. 7. |
| ∴ e. 2. 3. | 4 | $AD\,q + BD\,q + \overline{BC \times DC} \times 2 = AC\,q + BC\,q$. *Sub.* |
| Lastly | 5 | $AB\,q = AD\,q + BD\,q$. 47. 1. |
| ∴ e. 4. 5. | 6 | $\mathfrak{A}\mathfrak{B}q + \overline{\mathfrak{B}\mathfrak{C} \times \mathfrak{D}\mathfrak{C}} \times 2 = \mathfrak{A}\mathfrak{C}q + \mathfrak{B}\mathfrak{C}q$. *Sub.* Q. E. D. |

## PROP. XIV.

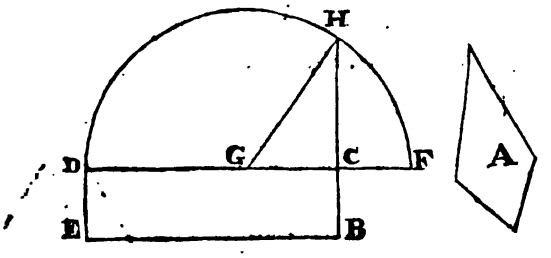
*To find a Right-line* CH, *whose Square is equal to a Right-lined Figure given* A.

## PREPARATION.

Make (45. 1.) the Rectangle DB = A, and produce the greater Side thereof DC to F, so that CF be = BC. Biſect (10. .) DF in G, about which as the Center at the Diſtance of GF, deſcribe the Circle FHD, and drawing out BC, till it touch the Circumference in H, joyn GH.

I ſay, $\mathfrak{C}\mathfrak{H}q = \mathfrak{A}$.

## DEMONSTRATION.




DC

| | | |
|---|---|---|
| | 1 | $DC \times CF + GC\,q = GF\,q$. 5. |
| And | 2 | $CF = BC$. *Conſt.* |
| ∴ e. 1. 2. | 3 | $DC \times BC + GC\,q = GF\,q$. *Sub.* |
| i. e. | 4 | $DB + GC\,q = GF\,q$. |
| Again | 5 | $GH = GF$. *Def.* 16. 1. |
| 5 & 2 | 6 | $GH\,q = GF\,q$. *Ax.* 9. |
| ∴ e. 4. 6. | 7 | $DB + GC\,q = GH\,q$. *Ax.* 1. |
| Alſo | 8 | $GC\,q + CH\,q = GH\,q$. 47. 1. |
| ∴ e. 7. 8. | 9 | $DB + GC\,q = GC\,q + CH\,q$. *Ax.* 1. |
| 9 — $GC\,q$. | 10 | $DB = CH\,q$. *Ax.* 12. |
| Alſo | 11 | $DB = A$. *Conſt.* |
| ∴ e. 10. 11. | 12 | $CH\,q = A$. *Ax.* 1. Q. E. F. & D. |

*The End of the Second Book.*

# EUCLID's ELEMENTS,

Demonſtrated after a New, Plain, and Eaſie METHOD.

## BOOK III.

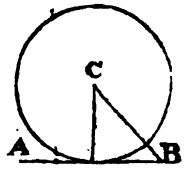
### DEFINITIONS.

1 **Equal Circles** are ſuch, whoſe Diameters or Semi-diameters are equal.

2 **A Line is ſaid to touch a Circle,** when, meeting with its Circumference, it does not cut it.

As the Line A B.

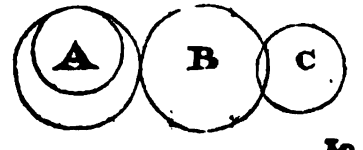
The Right-line C B, **cuts** the Circle.




3 **Circles touch,** when meeting they do not cut each other.

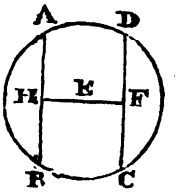
As the Circles A, B.

The Circle C, **cuts** the Circle B.



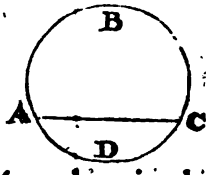

In

4 In a Circle ABCD, **Right-lines AB, DC, are said to be equally distant from the Center,** when Perpendiculars EF, EH, drawn from the Center E to them, are equal.



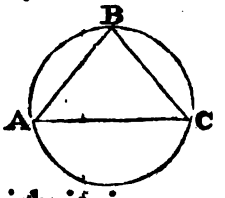

5 A **Segment of a Circle,** is a Figure terminated on one Side by a Right-line, and on the other by the Circumference of a Circle.

As ABC, ADC.



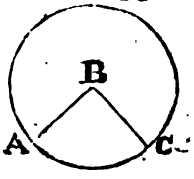

6 **An Angle is said to be in a Segment,** when in the Circumference thereof, some Point B is taken, and from it Right-lines BA, BC, are drawn to its Base AC.

As the Angle ABC.




7 **An Angle is upon that Arch,** to which it is opposed, or which is as its Base.

As the Angle ABC, is upon the Arch AC.




8 A **Sector** is a Figure contained under two Semidiameters, and the Arch which serves them for a Base.

As the Figure BCAB.

9 **Similar Segments,** are those that include equal Angles; or, in whom the Angles ABC, DEF, are equal.

As the Segments ABC, DEF.

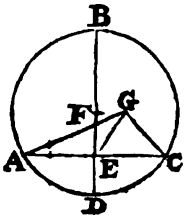
## PROPOSITION I.

*To find the Center* F, *of a given Circle* ABC.

### PREPARATION.

Draw a Right-line AC, anywiſe in the Circle ABC, which (10. 1.) biſect in E; through E draw (11. 1.) a Perpendicular BD, and biſect the ſame in F, ſo will the Point F be the Center.

If the Center be ſaid not to be in BD, and conſequently not in its Center F, ſuppoſe it in the Point G without BD; and draw GA, GE, GC; ſo (GC being *(Def.* 16. 1.) = GA, EC (*Conſt.*) = EA, and GE common) will ∠GEC be (8. 1.) = ∠GEA, and conſequently (*Def.* 10. 1.) Right.




I ſay, **F is the Center.**

### DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If the Center ſhould be in the Point G, without BD. |
| Then | 2 | GEC would be a r∠. |
| Alſo | 3 | FEC is a r∠. *Def.* 10. 1. |
| ∴ e. 2. 3. | 4 | ∠GEC would be = ∠FEC. *Ax.* 21. |
| But | 5 | ∠GEC ⌝ ∠FEC. *Ax.* 17. |
| ∴ | 6 | The Center is not in the Point G, without BD. |
| And | 7 | Since for the ſame Reaſon it is in no other Point without BD. |
| ∴ | 8 | The Center is in BD. |
| ∴ | 9 | **F is the Center.** Q. E. F. & D. |

COROL-

## COROLLARY.

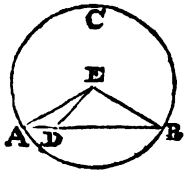
The Center of a Circle is in that Line, which falling perpendicularly upon another, divides it into two equal Parts.

## PROP. II.

*If in the Circumference of a Circle* ABC, *any two Points* A, B, *be taken, the Right-line* AB, *which joins those two Points, shall fall within the Circle.*

## PREPARATION.

Take, in the Right-line AB, any Point D; and from the Center E, draw EA, ED, EB.




I say, **AB falls within the Circle.**

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | EA = EB. *Def.* 16. 1. |
| ∴ e. 1. | 2 | ∠A = ∠B. 5. 1. |
| Again | 3 | ∠EDB ⊏ ∠A. 16. 1. |
| ∴ e. 2. 3. | 4 | ∠EDB ⊏ ∠B. *Ax.* 2. |
| ∴ e. 4. | 5 | EB ⊏ ED. 19. 1. |
| But | 6 | EB only reaches the Circumference. |
| ∴ e. 5. 6. | 7 | ED comes not so far. |
| ∴ e. 7. | 8 | The Point D falls within the Circle. |
| ∴ e. 8. | 9 | **AB falls within the Circle.** Q. E. D. |

## COROLLARY.

A Right-line doth not touch the Circumference of a Circle, but in one Point; because if it should touch it in two, it might be drawn from one of those Points to the other, and so would enter within the Circle, and consequently cut its Circumference, and not touch it.

## PROP. III.

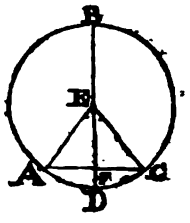
*If in a Circle* ABCD, *a Right-line* BD, *drawn thro' the Center, biſect any other Line* AC, *not drawn thro' the Center* E, *it ſhall cut it at Right-angles; and if it cut it at Right-angles, it ſhall alſo biſect the ſame.*

*Hyp.* 1 Suppoſe BD to biſect AC.

### PREPARATION.

Draw EA, EC.

I ſay, ∢EFA, ∢EFC, are rt ∠s.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | FA = FC. *Hyp.* |
| And | 2 | EA = EC. *Def.* 16. 1. |
| And | 3 | EF is common. |
| ∵ e. 1. 2. 3. | 4 | FA = FC, EA = EC, EF is common. |
| ∵ e. 4. | 5 | ∠EFA = ∠EFC. 8. 1. |
| ∵ e. 5. | 6 | ∢EFA, ∢EFC, are rt ∠s. *Def.* 10. 1. Q. E. D. |

*Hyp.* 2 Suppoſe BD to cut AC at Right-angles.

I ſay, FA = FC.

DEMON-

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\angle EFA = \angle EFC$. *Ax.* 21. |
| And | 2 | $EA = EC$. *Def.* 16. 1. |
| ∵ e. 2. | 3 | $\angle A = \angle C$. 5. 1. |
| Again | 4 | EF is common. |
| ∵ e. 1. 3. 4. | 5 | $\angle EFA = \angle EFC$, $\angle A = \angle C$, EF is common. |
| ∵ e. 5. | 6 | $\mathfrak{F}A = \mathfrak{F}C$. 26. 1. Q. E. D. |

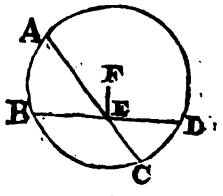
## PROP. IV.

*If in a Circle* ABCD, *two Right-lines* AC, BD, *not drawn thro' the Center* F, *cut each other in a Point* E, *those two Lines* AC, BD, *do not cut each other into two equal Parts.*

## PREPARATION.

Join EF.

I say, $\mathfrak{A}C$, $\mathfrak{B}D$, do not cut each other into two equal Parts.




## DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If AC, BD, should cut each other into two equal Parts. |
| Then | 2 | FEB, FEA, would be $r\angle$. 3. |
| ∵ e. 2, | 3 | $\angle FEB$ would be $= \angle FEA$. *Ax.* 21. |
| But | 4 | $\angle FEB \sqsubset \angle FEA$. *Ax.* 17. |
| ∵ | 5 | $\mathfrak{A}C$, $\mathfrak{B}D$, do not cut each other into two equal Parts. Q. E. D. |

## PROP. V.

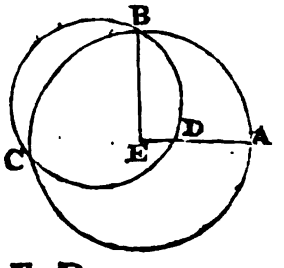
*If two Circles,* BAC, BDC, *cut one the other, they ſhall not have the ſame common Center.* E.

### PREPARATION.

From the Center E, of the Circle BAC, draw EA, EB.

I ſay, **E is not the common Center.**

### DEMONSTRATION.




| | | |
|---|---|---|
| For | 1 | If E ſhould be the common Center. |
| Then | 2 | EB would be = ED. *Def.* 16. 1. |
| Alſo | 3 | EB = EA. *Def.* 16. 1. |
| ∴ c. 3. 2. | 4 | EA would be = ED. *Ax.* 1. |
| But | 5 | EA ⊏ ED. *Ax.* 17. |
| ∴ | 6 | **E is not the common Center.** Q. E. D. |

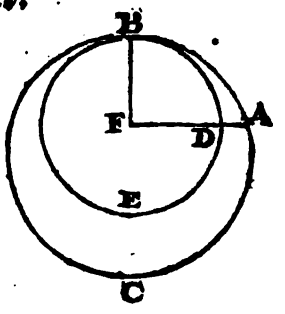
## PROP. VI.

*If two Circles,* BAC, BDE, *touch one another inwardly in* B, *they ſhall not have the ſame common Center* F.

### PREPARATION.

From the Center F, of the Circle BDE, draw FB, FD, and produce FD to A.

I say, ℱ is not the common Center.




DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If F should be the common Center. |
| Then | 2 | FB would be = FA. *Def.* 16. 1. |
| Also | 3 | FB = FD. *Def.* 16. 1. |
| ∴ e. 2. 3. | 4 | FA would be = FD. *Ax.* 1. |
| But | 5 | FA ⊏ FD. *Ax.* 17. |
| ∴ | 6 | ℱ is not the common Center. Q. E. D. |

## PROP. VII.

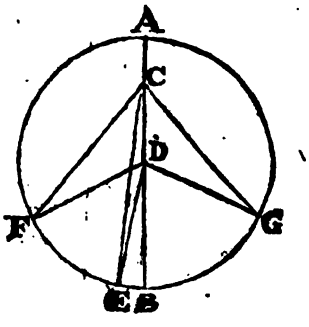
*If many Lines be drawn from any one Point* C, *within a Circle* AFBG, *which is not its Center* D, *to its Circumference*: 1st. *That Line* CB, *which passes thro' its Center, is the greatest*: 2d. *The Remainder of it* CA, *continued to the opposite Part of the Circumference, is the least*: 3d. *That Line* CE, *which is nearest the greatest* CB, *exceeds those that are more remote*: 4th. *There can be no more than two of them equal to each other.*

1st. *That Line* CB, *which passes thro' its Center* D, *is the greatest.*

PREPARATION.

Draw DE.

I say, for Example, ℭℬ ⊏ ℭℰ.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CD + DE ⊏ CE. 20. 1. |
| And | 2 | DB = DE. *Def.* 16. 1. |
| ∴ e. 1. 2. | 3 | CD + DB ⊏ CE. *Sub.* |
| i. e. | 4 | ℭℬ ⊏ ℭℰ. Q. E. D. |

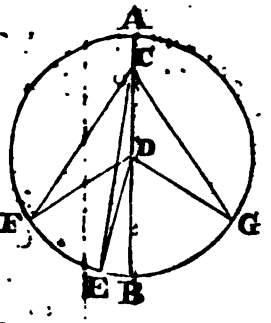
2d. *The Remainder of it* CA, *continued to the oppoſite Part of the Circumference, is the leaſt.*

## PREPARATION.

Draw CF, DF.

I ſay, for Example, ℂA ⌐ ℂF.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | DF ⌐ CD + CF. 20. 1. |
| And | 2 | DA = DF. *Def.* 16. 1. |
| ∴ e. 1. 2. | 3 | DA ⌐ CD + CF. *Sub.* |
| 3 — CD | 4 | DA — CD ⌐ CF. *Ax.* 16. |
| i. e. | 5 | ℂA ⌐ ℂF. Q. E. D. |

3d. *That Line* CE, *which is neareſt the greateſt* CB, *exceeds thoſe that are more remote.*

That is, for Example, ℂE ⌐ ℂF.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | DE = DF. *Def.* 16. 1. |
| And | 2 | CD is common. |
| And | 3 | ∠CDE ⌐ ∠CDF. *Ax.* 17. |
| ∴ e. 1. 2. 3. | 4 | DE = DF, CD is common, ∠CDE ⌐ ∠CDF. |
| ∴ e. 4. | 5 | ℂE ⌐ ℂF. 24. 1. Q. E. D. |

4th. *There can be no more than two of them equal to each other.*

## PREPARATION.

Make (23. 1.) at the Center D, ∠CDG = ∠CDF, and join CG.

I ſay,

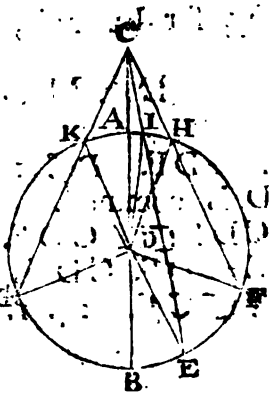
I say, **There can be no more than two of them equal to each other.**

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | DG = DF. *Def.* 16. 1. |
| And | 2 | CD is common, ∠CDG = ∠CDF. *Const.* |
| ∴ e. 1. 2. | 3 | DG = DF, CD is common, ∠CDG = ∠CDF. |
| ∴ e. 3. | 4 | CG = CF. 4. 1. |
| But | 5 | All other Lines will be either nearer or further off CB. |
| ∴ | 6 | All other Lines will be either ⊏ or ⊐ CG, and CF. 3d. 7. |
| ∴ | 7 | **There can be no more than two of them equal to each other.** Q. E. D. |

## PROP. VIII.

*If from a Point C, taken without a Circle AGBF, many Lines be drawn to its Circumference: 1st. Of all those that extend to the Concave Circumference, that Line CB, which passes thro' the Center D, is the greatest: 2d. That Line CE, which is nearest the greatest CB, exceeds those that are more remote: 3d. Among those that fall upon the Convex Circumference, that Line CA, which being continued passes thro' the Center, is the least: 4th. That Line CI, which is nearer CA, is less than those which are further off: 5th. There can be but two equal Lines drawn, from the same Point, either to the Concave or Convex Circumference.*



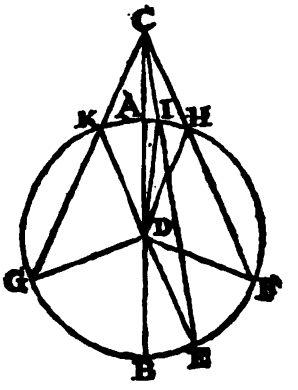

Of

18. *Of all those that extend to the Concave Circumference, that Line* CB, *which passes thro' the Center* D, *is the greatest.*

## PREPARATION.

Draw DE.

I say, for Example, ∢CDB ⊏ ∢CDE.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CD + DE ⊏ CE. 20. 1. |
| And | 2 | DB = DE. *Def.* 16. 1. |
| ∴ e. 1. 2. | 3 | CD + DB ⊏ CE. *Sub.* |
| i. e. | 4 | ∢CDB ⊏ ∢CDE. Q. E. D. |

19. *That Line* CE, *which is nearest the greatest* CB, *exceeds those that are more remote.*

## PREPARATION.

Draw CF, DF.

I say, for Example, ∢CDE ⊏ ∢CDF.

## DEMONSTRATION.

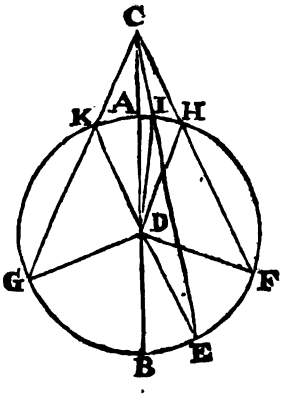
| | | |
|---|---|---|
| | 1. | DE = DF. *Def.* 16. 1. |
| And | 2 | CD is common. |
| And | 3 | ∠CDE ⊏ ∠CDF. *Ax.* 17. |
| ∴ e. 1. 2. 3. | 4 | DE = DF, CD is common, ∠CDE ⊏ ∠CDF. |
| ∴ e. 4. | 5 | ∢CDE ⊏ ∢CDF. 24. 1. Q. E. D. |

Among

3d. *Among those that fall upon the Convex Circumference, that Line* CA, *which being continued passes through the Center, is the least.*

PREPARATION.

Draw DI.




I say, for Example, ℂ𝔄 ⌝ ℂ𝔍.

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CD ⌝ CI + DI. 20. 1. |
| i. e. | 2 | CA + AD ⌝ CI + DI. |
| Again | 3 | DI = AD. *Def.* 16. 1. |
| ∵ e. 2. 3. | 4 | CA + AD ⌝ CI + AD. *Sub.* |
| 4 — AD | 5 | ℂ𝔄 ⌝ ℂ𝔍. *Ax.* 16. Q. E. D. |

4th. *That Line* CI, *which is nearer* CA, *is less than those which are further off.*

PREPARATION.

Draw DH.

I say, for Example, ℂ𝔍 ⌝ ℂ𝔇.

DEMONSTRATION.

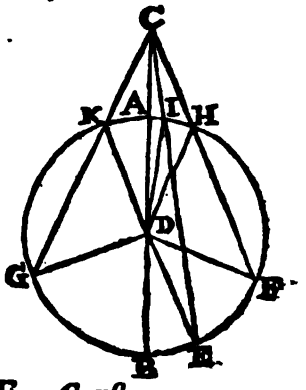
| | | |
|---|---|---|
| | 1 | CI + DI ⌝ CH + DH. 21. 1. |
| And | 2 | DI = DH. *Def.* 16. 1. |
| ∵ e. 1. 2. | 3 | CI + DH ⌝ CH + DH. *Sub.* |
| 3 — DH | 4 | ℂ𝔍 ⌝ ℂ𝔇. *Ax.* 16. Q. E. D. |

5th. *There can be but two equal Lines drawn, from the same Point, either to the Concave or Convex Circumference.*

## PREPARATION.

Make (23. 1.) at the Center D, ∠CDG = ∠CDF, ∠CDK = ∠CDH, and draw CKG.

I say, **There can be but two equal Lines drawn, from the same Point, either to the Concave or Conver Circumference.**




## DEMONSTRATION.

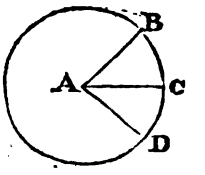
| | | |
|---|---|---|
| | 1 | DG = DF. *Def.* 16. 1. |
| And | 2 | CD is common, ∠CDG = ∠CDF. *Const.* |
| ∴ 1. 2. | 3 | DG = DF, CD is common, ∠CDG = ∠CDF. |
| ∴ e. 3. | 4 | CG = CF. 4. 1. |
| Again | 5 | DK = DH. *Def.* 16. 1. |
| And | 6 | CD is common, ∠CDK = ∠CDH. *Const.* |
| ∴ e. 5. 6. | 7 | DK = DH, CD is common, ∠CDK = ∠CDH. |
| ∴ e. 7. | 8 | CK = CH. 4. 1. |
| ∴ 4. 8. | 9 | CG = CF, CK = CH. |
| But | 10 | All other Lines will be either nearer or more remote from CB and CA. |
| ∴ | 11 | All other Lines will be either ⊏ or ⊐ CG and CF, CK and CH 2d. and 4th. 8. |
| ∴ | 12 | **There can be but to equal Lines drawn, from the same Point, either to the Concave or Conver Circumference.** Q. E. D. |

## PROP. IX.

*That Point, from whence three equal Lines* AB, AC, AD, *can be drawn to the Circumference of a Circle* ABCD, *is its Center.*

That

That is, **A is its Center.**




DEMONSTRATION.

1 More than two equal Right-lines cannot be drawn from a Point out of the Center to the Circumference. 4th. 7.

But 2 AB = AC = AD. *Hyp.*

∴ 3 **A is its Center.** Q. E. D.

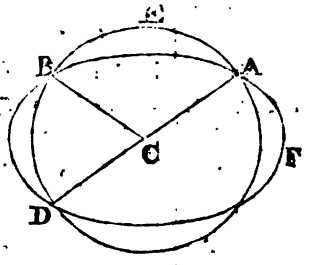
## PROP. X.

*Two Circles* AEBD, ABDF, *cannot cut each other in more than two Points.*

## PREPARATION.

If AEBD, ABDF, be said to cut each other in more than two Points A, B, D; find (1) the Center C of the Circle AEBD; and draw CA, CB, CD, which (*Def.* 16. 1.) will be equal the one to the other.

I say, **AEBD, ABDF, cannot cut each other in more than two Points.**




DEMONSTRATION.

For 1 If AEBD, ABDF, should cut each other in more than two Points.

Then 2 C would be the Center of the Circle ABDF. 9.

Also 3 C is the Center of the Circle AEBD.

∴ e. 2. 3. 4 Two Circles which cut each other, would have the same common Center.

But 5 Two Circles which cut each other, have not the same common Center. 5.

∴ 6 **AEBD, ABDF, cannot cut each other in more than two Points.** Q. E. D.

## PROP. XI.

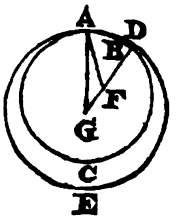
*If two Circles, ABC, ADE, touch each other within, at the Point A, the Right-line drawn thro' their Centers, being produced, will pass thro' the Point of Contact.*

### PREPARATION.

Draw GA, GD, FA.

I say, **The Right-line drawn thro' their Center will pass thro' the Point of Contact.**

### DEMONSTRATION.




If the Points G, F, were their Centers, so that
For 1 the Right-line drawn thro' them should pass thro' other Points.
Then 2 GA would be = GD, FA would be = FB. *Def.* 16. 1.
Now 3 GA ⌝ GF + FA. 20. 1.
∴ 2. 3. 4 GD would be ⌝ GF + FB. *Sub.*
But 5 GD ⌜ GF + FB. *Ax.* 17.
∴ 6 The Right-line drawn thro' their Centers will not pass thro' other Points.
∴ 7 **The Right-line drawn thro' their Centers will pass thro' the Point of Contact.** Q. E. D.

PROP.

## PROP. XII.

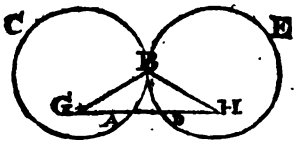
*If two Circles* ABC, DBE, *touch each other without, at the Point* B, *the Right-line drawn thro' their Centers will pass thro' the Point of Contact.*

## PREPARATION.

Draw GB, HB, GH.

I say, **The Right-line drawn thro' their Centers will pass thro' the Point of Contact.**

## DEMONSTRATION.




For 1 If the Points G, H, were their Centers, so that the Right-line drawn thro' them should pass thro' other Points.

Then 2 GA would be = GB, HD would be = HB. *Def.* 16. 1.

Now 3 GH ⌝ GB + HB. 20. 1.

∴ c. 2. 3 4 GH would be ⌝ GA + HD. *Sub.*

But 5 GH ⌐ GA + HD. *Ax.* 17.

∴ 6 The Right-line drawn thro' their Centers will not pass thro' other Points.

∴ 7 **The Right-line drawn thro' their Centers will pass thro' the Point of Contact.** Q. E. D.

## PROP. XIII.

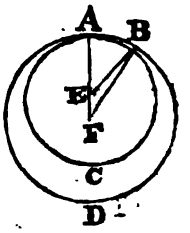
*Two Circles* ABC, ABD, *touch each other only in one Point, whether it be within or without.*

*Hyp.* 1 Suppoſe them to touch within, at the Point A.

### PREPARATION.

Having drawn thro' their Centers E, F, the Right-line EF, which being produced, will (11.) paſs thro' the Point of Contact; draw the Right-lines EB, FB.

I ſay, **They touch only in one Point.**




### DEMONSTRATION.

For 1 If they ſhould touch in another Point; as B.
Then 2 EA would be = EB. *Def.* 16. 1.
Now 3 EB + EF ⊏ FB. 20. 1.
∴ e. 2. 3. 4 EA + EF would be ⊏ FB. *Sub.*
But 5 EA + EF = FB. *Def.* 16. 1.
∴ 6 They do not touch in another Point; as B.
∴ 7 **They touch only in one Point.** Q. E. D.

*Hyp.* 2 Suppoſe them to touch without, at the Point A.

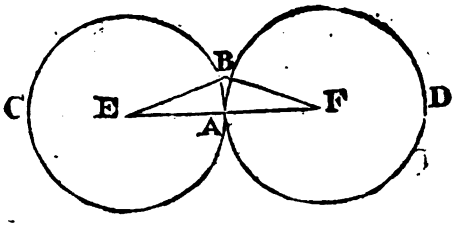
PRE-

## PREPARATION.

Having drawn thro' their Centers E, F, the Right-line EF, which ( 12. ) will pass thro' the Point of Contact; draw the Right-lines EB, FB.

I say, **They touch only in one Point.**

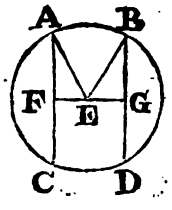
## DEMONSTRATION.




| | | |
|---|---|---|
| For | 1 | If they should touch in another Point; as B. |
| Then | 2 | EA would be = EB, FA would be = FB. *Def.* 16. 1. |
| Now | 3 | EB + FB ⊏ EF. 20. 1. |
| ∴ e. 2. 3. | 4 | EA + FA would be ⊏ EF. *Sub.* |
| But | 5 | EA + FA = EF. |
| ∴ | 6 | They do not touch in another Point; as B. |
| ∴ | 7 | **They touch only in one Point.** Q. E. D. |

## PROP. XIV.

*In a Circle* ABDC, *equal Right-lines* AC, BD, *are equally distant from the Center; and Right-lines* AC, BD, *equally distant from the Center, are equal.*




Suppose

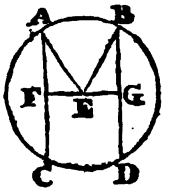
| | | |
|---|---|---|
| *Hyp.* | 1 | Suppose AC = BD. |

## PREPARATION.

From the Center E, draw the Perpendiculars EF, EG, which (3.) will bisect the Lines AC, BD, and join EA, EB.

I say, AC, BD, are equally distant from the Center.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AC = BD. *Hyp.* |
| ∴ c. 1. | 2 | AF = BG. *Ax.* 8. |
| Again | 3 | EA = EB. *Def.* 16. 1. |
| 2 ⊙. 2. | 4 | $AF^q = BG^q$. *Ax.* 9. |
| 3 ⊙. 2. | 5 | $EA^q = EB^q$. *Ax.* 9. |
| 5 — 4. | 6 | $EA^q - AF^q = EB^q - BG^q$. *Ax.* 12. |
| i. c. | 7 | $EF^q = EG^q$. 47. 1. |
| 7 in 2. | 8 | EF = EG. *Ax.* 10. |
| ∴ c. 8. | 9 | AC, BD, are equally distant from the Center. *Def.* 4. Q. E. D. |

| | | |
|---|---|---|
| *Hyp.* | 2 | Suppose AG, BD, equally distant from the Center. |

I say, AC = BD.

DEMON-

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $EF = EG$. *Def.* 4. |
| And | 2 | $EA = EB$. *Def.* 16. 1. |
| 1. ⊙. 2. | 3 | $EF^q = EG^q$. *Ax.* 9. |
| 2. ⊙. 2. | 4 | $EA^q = EB^q$. *Ax.* 9. |
| 4. — 3. | 5 | $EA^q - EF^q = EB^q - EG^q$. *Ax.* 12. |
| i. e. | 6 | $AF^q = BG^q$. 47. 1. |
| 6. *inv.* 2. | 7 | $AF = BG$. *Ax.* 10. |
| ∴ e. 7. | 8 | $\mathfrak{AC} = \mathfrak{BD}$. *Ax.* 7. Q. E. D. |

## PROP. XV.

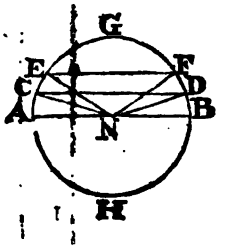
*If several Right-lines* AB, CD, EF, *be drawn in a Circle* AHBG, *the greatest is the Diameter* AB, *and that which is nearest the Center* N, *is greater than that which is further off.*

1 *The greatest is the Diameter* AB.

## PREPARATION.

Draw NC, ND.

I say, for Example, $\mathfrak{AB} \sqsubset \mathfrak{CD}$.



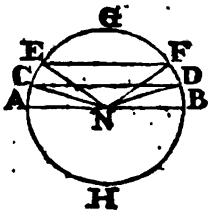

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $NC = NA$. *Def.* 16. 1. |
| And | 2 | $ND = NB$. *Def.* 16. 1. |
| 1 + 2. | 3 | $NC + ND = NA + NB$. *Ax.* 11. |
| Again | 4 | $NC + ND \sqsubset CD$. 20. 1. |
| ∴ e. 3. 4. | 5 | $NA + NB \sqsubset CD$. *Ax.* 3. |
| i. e. | 6 | $\mathfrak{AB} \sqsubset \mathfrak{CD}$. Q. E. D. |

2 *That which is nearest the Center, is greater than that which is further off.*

## PREPARATION.

Draw NE, NF.




I say, for Example, ℂ𝔻 ⊏ ℂ𝔽.

## DEMONSTRATION.

1 NC = NE, ND = NF. *Def.* 16. 1.
And 2 ∠CND ⊏ ∠ENF. *Ax.* 17.
∴ e. 1. 2. 3 NC = NE, ND = NF, ∠CND ⊏ ∠ENF.
∴ e. 3. 4 ℂ𝔻 ⊏ ℂ𝔽; 24. 1. Q. E. D.

## PROP. XVI.

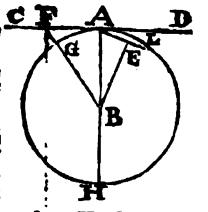
*A Perpendicular Line* CD, *drawn thro' the Extremity of the Diameter of a Circle* ALHG, *is wholly without the Circle; and any other Right-line* AL, *drawn from the Point of Contact next the Circumference, falls within the Circle.*

## PREPARATION.

1 Draw BF.

I say, ℂ𝔻 is wholly without the Circle.

DEMON-




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | FAB is a r∠. *Def.* 10. 1. |
| ∴ e. 1. | 2 | ∠FAB ⊏ ∠AFB. 32. 1. |
| ∴ e. 2. | 3 | BF ⊏ BA. 19. 1. |
| ∴ e. 3. | 4 | The Point F is further from the Center than the Point A. |
| Now | 5 | The Point A is in the Circumference. |
| ∴ | 6 | The Point F is without the Circumference. |
| And | 7 | Since for the same Reason every other Point in CD is without the Circumference, besides A. |
| ∴ | 8 | **CD is wholly without the Circle.** Q. E. D. |

## PREPARATION.

| | | |
|---|---|---|
| | 2 | Draw BE ⊥ AL. |

I say, **AL falls within the Circle.**

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | BEA is a r∠. *Def.* 10. 1. |
| ∴ e. 1. | 2 | ∠BEA ⊏ ∠BAE. 32. 1. |
| ∴ e. 2. | 3 | BA ⊏ BE. 19. 1. |
| ∴ e. 3. | 4 | The Point E is nearer the Center than the Point A. |
| Now | 5 | The Point A is in the Circumference. |
| ∴ | 6 | The Point E is within the Circumference. |
| ∴ | 7 | **AL falls within the Circle.** Q. E. D. |

## COROLLARY.

A Right-line drawn thro' the Extremity of the Diameter of a Circle, and at Right-angles, is, (*Def.* 2.) a Tangent to the said Circle.

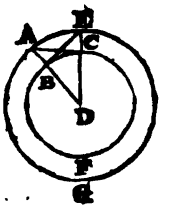
## PROP. XVII.

*From a Point given* A, *to draw a Right-line* AC, *which ſhall touch a Circle given* DCBF.

### PREPARATION.

From the Center D, of the Circle DCBF, let a Line DA, cutting the Circumference in B, be drawn to the given Point A; from the Center D, at the Diſtance DA, deſcribe another Circle DAEG, and from B, draw a Perpendicular to AD, which will meet the Circle DAEG, in the Point E; then having drawn ED, meeting with the Circle DCBF, in the Point C, draw AC.

I ſay, AC touches the Circle DCBF.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | DB = DC, DE = DA. *Def.* 16. 1. |
| And | 2 | ∠D is common. |
| ∴ e. 1. 2. | 3 | DB = DC, DE = DA, ∠D is common. |
| ∴ e. 3. | 4 | ∠DBE = ∠DCA. 4. 1. |
| But | 5 | DBE is a r∠. *Def.* 10. 1. |
| ∴ | 6 | DCA is a r∠. |
| ∴ e. 6. | 7 | AC touches the Circle DCBF. *Cor.* 16. Q.E.D. |

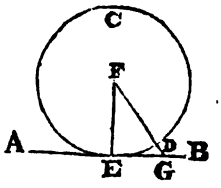
## PROP. XVIII.

*If a Right-line* AB, *touch a Circle* FEDC, *and from the Center* F, *to the Point of Contact* E, *a Right-line* FE *be drawn; that Line* FE, *is perpendicular to the Tangent* AB.

PRE-

## PREPARATION.

If FE be said not to be ⊥ AB, let any other Line FG be ⊥ AB, so (∠ FGE being (*Def.* 10. 1.) Right, and consequently (32. 1.) ⊏ ∠ FEG) will FE be (19. 1.) ⊏ that Line FG.




I say, **FE ⊥ AB.**

## DEMONSTRATION.

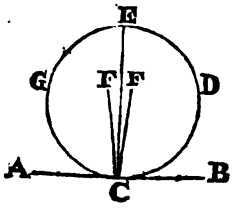
| | | |
|---|---|---|
| For | 1 | If any other Line, besides FE, should be ⊥ AB; as FG. |
| Then | 2 | FE would be ⊏ that Line FG. |
| ∴ c. 2. | 3 | AB would cut the Circumference. |
| But | 4 | AB does not cut the Circumference. *Def.* 2. |
| ∴ | 5 | No other Line, besides FE, is ⊥ AB. |
| ∴ | 6 | **FE ⊥ AB.** Q. E. D. |

## PROP. XIX.

*If a Right-line* AB, *touch a Circle* CDEG, *and from the Point of Contact* C, *a Right-line* CE, *be erected at Right-angles to the Tangent, the Center of the Circle is in the Right-line* CE *so erected.*

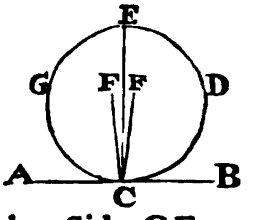
## PREPARATION.

If the Center be said not to be in CE, let it be in any Line CF, on either Side CE, so (CF being (18.) ⊥ AB, and consequently (*Def.* 10. 1.) ∠ FCB Right) will ∠ FCB, form'd by that Line and the Tangent CB on either Side CE; be (*Ax.* 21.) = ∠ ECB.




I say, **The Center is in CE.**

## DEMONSTRATION.




For 1 If the Center ſhould be in any Line CF, on either Side CE.

Then 2 ∠ FCB, formed by that Line and the Tangent CB, would be = ∠ ECB.

But 3 ∠ FCB is either ⌐ or ¬ ∠ ECB. *Ax.* 17.

∴ 4 The Center is not in any Line CF, on either Side CE.

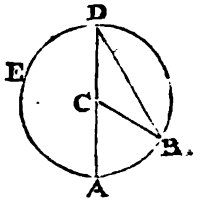
∴ 5 **The Center is in CE.** Q. E. D.

## PROP. XX.

*An Angle at the Center of a Circle* ABDE, *is always double to the Angle at the Circumference, when both the Angles ſtand upon the ſame Arch.*

*Caſe* 1 If one of the Sides DA, of the Angle at the Circumference D, be in a Right-line with one of the Sides CA, of the Angle at the Center BCA.

I ſay, ∠ BCA = 2 ∠ D.




## DEMONSTRATION.

1 CB = CD. *Def.* 16. 1.

∴ e. 1. 2. ∠ B = ∠ D. 5. 1.

Again 3 ∠ BCA = ∠ B + ∠ D. 32. 1.

∴ e 2. 3. 4 ∠ BCA = ∠ D + ∠ D. *Sub.*

i. e. 5 ∠ BCA = 2 ∠ D. Q. E. D.

*Caſe* 2 If the two Sides DB, DF, of the Angle at the Circumference BDF, incloſe the Angle at the Center BCF.

PRE-

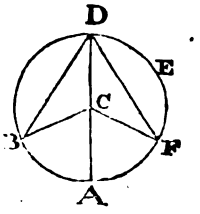
PREPARATION.

Draw the Diameter DA.

I say, ∠ BCF = 2 ∠ BDF.

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | ∠BCA = 2 ∠BDA. *Case* 1. |
| And | 2 | ∠FCA = 2 ∠FDA. *Case* 1. |
| 1 + 2 | 3 | ∠BCA + ∠FCA = 2 ∠BDA + 2 ∠FDA. *Ax.* 11. |
| Also | 4 | ∠BCA + ∠FCA = ∠BCF. *Ax.* 18. |
| ∵ e. 4. 3. | 5 | ∠BCF = 2 ∠BDA + 2 ∠FDA. *Ax.* 1. |
| Again | 6 | ∠BDF = ∠BDA + ∠FDA *Ax.* 18. |
| 6 × 2 | 7 | 2 ∠BDF = 2 ∠BDA + 2 ∠FDA. *Ax.* 13. |
| ∵ e. 5. 7. | 8 | ∠BCF = 2 ∠BDF. *Ax.* 1. Q. E. D. |




*Case* 3 If one of the Sides DB, of the Angle at the Circumference FDB, cut one of the Sides CF, of the Angle at the Center FCB.

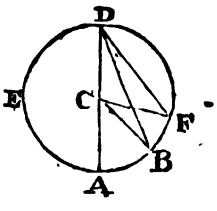
PREPARATION.

From the angular Point D, draw the Diameter DA.

I say, ∠ FCB = 2 ∠ FDB.

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | ∠FCA = 2 ∠FDA. *Case* 1. |
| And | 2 | ∠BCA = 2 ∠BDA. *Case* 1. |
| 1 — 2 | 3 | ∠FCA — ∠BCA = 2 ∠FDA — 2 ∠BDA. *Ax.* 12. |
| Also | 4 | ∠FCA — ∠BCA = ∠FCB. |
| ∵ e. 4. 3. | 5 | ∠FCB = 2 ∠FDA — 2 ∠BDA. *Ax.* 1. |
| Again | 6 | ∠FDB = ∠FDA — ∠BDA. |
| 6 × 2 | 7 | 2 ∠FDB = 2 ∠FDA — 2 ∠BDA. *Ax.* 13. |
| ∵ e. 5. 7. | 8 | ∠FCB = 2 ∠FDB. *Ax.* 1. Q. E. D. |



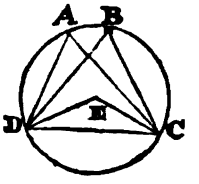
## PROP. XXI.

*In a Circle* ABCD, *the Angles* DAC, DBC, *which are in the same Segment, are equal.*

*Case* 1 If the Segment ABCD be greater than a Semicircle.

### PREPARATION.

From the Center E, draw EC, ED.




I say, ∠ DAC = ∠ DBC.

### DEMONSTRATION.

1 ∠ E = 2 ∠ DAC. 20.
Also 2 ∠ E = 2 ∠ DBC. 20.
∵ c. 1. 2. 3 2 ∠ DAC = 2 ∠ DBC. *Ax.* 1.
3 ÷ 2 4 ∠ DAC = ∠ DBC. *Ax.* 14. Q. E. D.

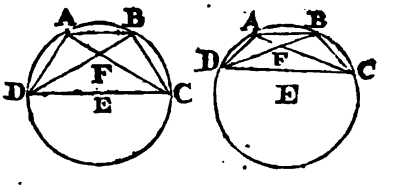
*Case* 2 If the Segment ABCD, be equal to, or less than a Semicircle.

### PREPARATION.

Draw AB.

I say, ∠ DAC = ∠ DBC.

### DEMONSTRATION.




1 ∠ ADB = ∠ BCA. *Case* 1.
i. e. 2 ∠ ADF = ∠ BCF.
Again 3 ∠ DFA = ∠ CFB. 15. 1.
And 4 ∠ DFA + ∠ ADF + ∠ DAF = ∠ CFB + ∠ BCF + ∠ CBF. *Cor.* 1. 32. 1.
3 + 2 5 ∠ DFA + ∠ ADF = ∠ CFB + ∠ BCF. *Ax.* 11.
4 — 5 6 ∠ DAF = ∠ CBF. *Ax.* 12.
i. e. 7 ∠ DAC = ∠ DBC. Q. E. D.

PROP.

## PROP. XXII.

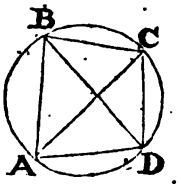
*The two oppoſite Angles of a quadrilateral Figure inſcribed in a Circle, are taken together, equal to two Right-Angles.*

### PREPARATION.

Draw AC, BD.

I ſay, $\angle ABC + \angle ADC$, or, $\angle BCD + \angle BAD = 2r\angle s$.

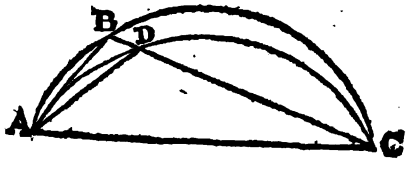
### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $\angle BDA = \angle BCA$. 21. |
| And | 2 | $\angle BDC = \angle BAC$. 21. |
| 1 + 2 | 3 | $\angle BDA + \angle BDC = \angle BCA + \angle BAC$. *Ax.* 11. |
| i. e. | 4 | $\angle ADC = \angle BCA + \angle BAC$. |
| Again | 5 | $\angle ABC + \angle BCA + \angle BAC = 2r\angle s$. 32. 1. |
| ∴ e. 4. 5. | 6 | $\angle ABC + \angle ADC = 2r\angle s$. *Sub.* |
| Laſtly | 7 | $\angle ABC + \angle ADC + \angle BCD + \angle BAD = 4r\angle s$. *Cor.* 2. *The.* 1. 1. |
| 7 — 6 | 8 | $\angle BCD + \angle BAD = 2r\angle s$. *Ax.* 12 |
| ∴ e. 6. 8. | 9 | $\angle ABC + \angle ADC$, or, $\angle BCD + \angle BAD = 2r\angle s$. Q. E. D. |

## PROP. XXIII.

*Two Similar and unequal Segments of Circles, ABC, ADC, cannot be ſet on the ſame Right-line AC, and the ſame Side thereof.*



## PREPARATION.

Draw CB, cutting the Circumferences in D and B, and join AB, AD.

I say, **ABC, ADC, are not similar.**

## DEMONSTRATION.

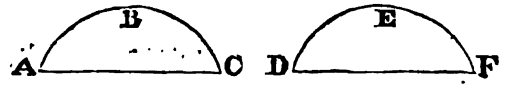
| | | |
|---|---|---|
| For | 1 | If ABC, ADC, were similar. |
| Then | 2 | ∠ADC would be = ∠ABC. *Def.* 9. |
| But | 3 | ∠ADC ⊏ ∠ABC. 16. 1. |
| ∴ | 4 | **ABC, ADC, are not similar.** Q. E. D. |

## PROP. XXIV.

*Similar Segments of Circles,* ABC, DEF, *set upon equal Right-lines* AC, DF, *are equal one to the other.*

That is, **ABC = DEF.**

## DEMONSTRATION.




| | | |
|---|---|---|
| For | 1 | If ABC be placed on DEF. |
| Then | 2 | AC being = DF. *Hyp.* |
| ∴ e. 2. | 3 | The Point A will fall on D, the Point C on F. |
| ∴ e. 3. | 4 | ABC, DEF, will stand on the same Right-line, and the same side thereof. |
| ∴ e. 4. | 5 | **ABC = DEF.** 23. Q. E. D. |

PROP.

## PROP. XXV.

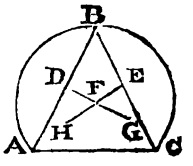
*The Segment* ABC *of a Circle being given, to find the Center of that Circle.*

### PREPARATION.

Chooſe at Pleaſure three Points A, B, C, upon the Circumference ABC, and having joined the Right-lines BA, BC, and biſected them at the Points D, E, erect on thoſe Points the Perpendiculars DG, EH.

I ſay, **F is the Center.**

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | DG biſects BA, EH biſects BC at r∠s. *Conſt.* |
| ∴ e. 1. | 2 | The Center is in DG and EH. *Cor.* 1. |
| ∴ e. 2. | 3 | The Center is in their common Interſection. |
| ∴ e. 3. | 4 | **F is the Center.** Q. E. F. & D. |

## PROP. XXVI.

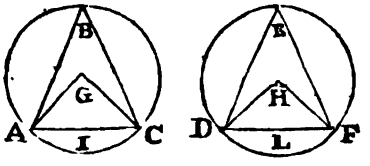
*In equal Circles,* AICB, DLFE, *equal Angles ſtand upon equal Arches of the Circumferences,* AIC, DLF, *whether thoſe equal Angles be at the Centers, as* G, H, *or at the Circumferences, as* B, E.

## PREPARATION.

Join AC, DF.

I ſay, Arch AIC = Arch DLF.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | GA = HD, GC = HF. *Def.* 1. |
| And | 2 | ∠G = ∠H. *Hyp.* |
| ∴ e. 1. 2. | 3 | GA = HD, GC = HF, ∠G = ∠H. |
| ∴ e. 3. | 4 | AC = DF. 4. 1. |
| 2 ÷ 2 | 5 | ½ ∠G = ½ ∠H. *Ax.* 14. |
| Alſo | 6 | ½ ∠G = ∠B. 20. |
| ∴ e. 5. 6. | 7 | ½ ∠H = ∠B. *Ax.* 8. |
| Alſo | 8 | ½ ∠H = ∠E. 20. |
| ∴ e. 7. 8. | 9 | ∠B = ∠E. *Ax.* 8. |
| ∴ e. 9. | 10 | The Segments ABC, DEF, are ſimilar. *Def.* 9. |
| ∴ e. 10. 4. | 11 | The Segments ABC, DEF, are ſimilar, and ſtand on equal Lines AC, DF. |
| ∴ e. 11. | 12 | Segment ABC = Segment DEF. 24. |
| ∴ e. 12. | 13 | Arch ABC = Arch DEF. |
| Alſo | 14 | Arch ABC + Arch AIC = Arch DEF + Arch DLF. *Hyp.* |
| 14 — 13. | 15 | Arch AIC = Arch DLF. *Ax.* 12. Q. E. D. |

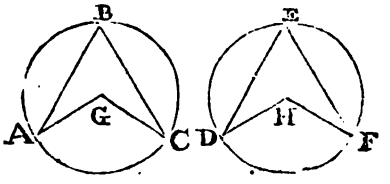
PROP.

## PROP. XXVII.

*In equal Circles,* ABC, DEF, *the Angles ſtanding upon equal Arches of the Circumferences* AC, DF, *are equal between themſelves, whether they be made at the Centers,* as G, H, *or at the Circumferences,* as B, E.

That is, $\angle \mathfrak{G} = \angle \mathfrak{H}$. $\angle \mathfrak{B} = \angle \mathfrak{E}$.

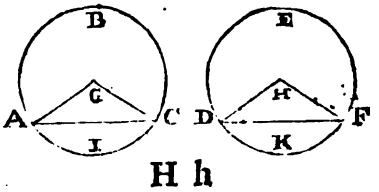
### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AC, DF, are equal Arches of equal Circles. *Hyp.* |
| ∵ e. 1. | 2 | The $\angle$'s which they meaſure are equal. |
| i. e. | 3 | $\angle G = \angle H$. |
| $3 \div 2$ | 4 | $\frac{1}{2} \angle G = \frac{1}{2} \angle H$. *Ax.* 14. |
| Alſo | 5 | $\angle E = \frac{1}{2} \angle H$. 20. |
| ∵ e. 4. 5. | 6 | $\frac{1}{2} \angle G = \angle E$. *Ax.* 8. |
| Alſo | 7 | $\frac{1}{2} \angle G = \angle B$. 20. |
| ∵ e. 7. 6. | 8 | $\angle B = \angle E$. *Ax.* 8. |
| ∵ e. 3. 8. | 9 | $\angle \mathfrak{G} = \angle \mathfrak{H}$. $\angle \mathfrak{B} = \angle \mathfrak{E}$. Q. E. D. |

## PROP. XXVIII.

*In equal Circles,* AICB, DKFE, *equal Right-lines* AC, DF, *cut off equal Arches of the Circumferences, the greateſt* ABC *equal to the greateſt* DEF, *and the leaſt* AIC, *equal to the leaſt* DKF.

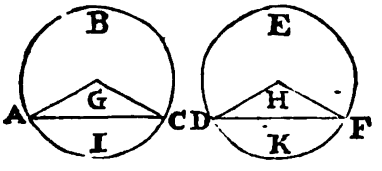


## PREPARATION.

From the Centers G, H, draw GA, GC, HD, HF.

I say, Arch ABC = Arch DEF. Arch AIC = Arch DKF.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | GA = HD, GC = HF. *Def.* 1. |
| And | 2 | AC = DF. *Hyp.* |
| ∵ e. 1. 2. | 3 | GA = HD, GC = HF, AC = DF. |
| ∵ e. 3. | 4 | ∠G = ∠H. 8. 1. |
| ∵ e. 4. | 5 | Arch AIC = Arch DKF. 26. |
| Also | 6 | Arch ABC + Arch AIC = Arch DEF + Arch DKF. *Hyp.* |
| 6 — 5 | 7 | Arch ABC = Arch DEF. *Ax.* 12. |
| ∵ e. 7. 5. | 8 | Arch ABC = Arch DEF. Arch AIC = Arch DKF. Q. E. D. |

## PROP. XXIX.

*In equal Circles*, AICB, DKFE, *equal Arches* AIC, DKF, *are subtended by equal Right-lines* AC, DF.

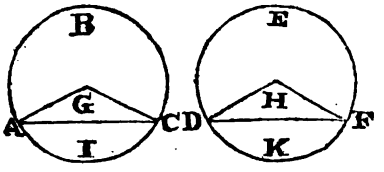
## PREPARATION.

From the Centers G, H, draw GA, GC, HD, HF, and join AC, DF.

I say, AC = DF.

DEMON-

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | GA = HD, GC = HF. *Def.* 1. |
| And | 2 | Arch AIC = Arch DKF. *Hyp.* |
| ∴ e. 2. | 3 | ∠G = ∠H. 27. |
| ∴ e. 1. 3. | 4 | GA = HD, GC = HF, ∠G = ∠H. |
| ∴ e. 4. | 5 | AC = DF. 4. 1. Q. E. D. |

Note. *This and the three precedent Propositions may be understood also of the same Circle.*

## PROP. XXX.

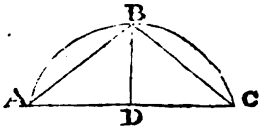
*To cut a Periphery given* ABC *into two equal Parts.*

## PREPARATION.

Draw the Right-line AC, and bisect it in D; from D draw a Perpendicular DB, meeting with the Arch in B, and join AB, CB.

I say, Arch AB = Arch CB.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | ADB, CDB. are r∠'s. *Def.* 10. 1. |
| ∴ e. 1. | 2 | ∠ADB = ∠CDB. *Ax.* 21. |
| Again | 3 | AD = DC, DB is common. *Const.* |
| ∴ e. 3. 2. | 4 | AD = DC, DB is common, ∠ADB = ∠CDB. |
| ∴ e. 4. | 5 | AB = CB. 4. 1. |
| ∴ e. 5. | 6 | Arch AB = Arch CB. 28. Q. E. F. & D. |

## PROP. XXXI.

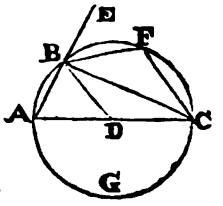
*In a Circle* ABFCG, *the Angle* ABC, *which is in the Semicircle* ABFC, *is a Right-angle; the Angle* BAC, *which is in the greater Segment* BAGC, *is Acute, or leſs than a Right-angle; and the Angle* BFC, *which is in the leſſer Segment* BFC, *is Obtuſe, or greater than a Right-angle.*

### PREPARATION.

From the Center D, draw DB, and produce AB to E.

I ſay, ∠ABC is Right. ∠BAC Acute. ∠BFC Obtuſe.

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | DA = DB. *Def.* 16. 1. |
| ∵ e. 1. | 2 | ∠A = ∠DBA. 5. 1. |
| Again | 3 | DC = DB. *Def.* 16. 1. |
| ∵ e. 3. | 4 | ∠DCB = ∠DBC. 5. 1. |
| 2 + 4. | 5 | ∠A + ∠DCB = ∠DBA + ∠DBC. *Ax.* 11. |
| Alſo | 6 | ∠A + ∠DCB = ∠EBC. 32. 1. |
| ∵ e. 5. 6. | 7 | ∠DBA + ∠DBC = ∠EBC. *Ax.* 1. |
| i. e. | 8 | ∠ABC = ∠EBC. |
| ∵ e. 8. | 9 | ABC is a r∠. *Def.* 10. 1. |
| ∵ e. 9. | 10 | ∠BAC is Acute. *Cor.* 1. 17. 1. |
| Laſtly | 11 | ∠BAC + ∠BFC = 2r∠s. 22. |
| But e. 10. | 12 | ∠BAC is Acute, or ⌝ a r∠. |
| ∵ e. 11. 12. | 13 | ∠BFC is Obtuſe, or ⌜ a r∠. |
| ∵ e. 9. 10. 13. | 14 | ∠ABC is Right. ∠BAC Acute. ∠BFC Obtuſe. Q. E. D. |

PROP.

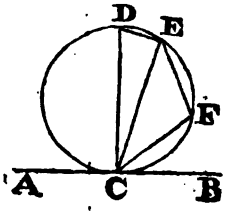
## PROP. XXXII.

*If a Right-line* AB *touch a Circle, and from the Point of Contact* C, *be drawn a Right-line* CE, *cutting the Circle, the Angles* ECB, ECA, *which it makes with the Tangent* AB, *are equal to thoſe Angles* EDC, EFC, *which are made in the alternate Segments of the Circle.*

### PREPARATION.

Let CD, the ſide of the Angle EDC, be perpendicular to AB, (for it is to the ſame Purpoſe, ſince (27.) all the Angles which ſtand on the Arch EFC are equal) and taking any Point F, in the Arch EC, join DE, EF, FC.

I ſay, $\angle ECB = \angle EDC$. $\angle ECA = \angle EFC$.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CD $\perp$ AB. *Conſt.* |
| ∵ e. 1. | 2 | The Center is in CD. 19. |
| ∵ e. 2. | 3 | CD is the Diameter. *Def.* 17. 1. |
| ∵ e. 3. | 4 | CFED is a Semicircle. |
| ∵ e. 4. | 5 | DEC is a r$\angle$. 31. |
| ∵ e. 5. | 6 | $\angle EDC + \angle ECD =$ r$\angle$. *Cor.* 3. 32. 1. |
| Again | 7 | DCB is a r$\angle$. *Def.* 10. 1. |
| ∵ e. 7. | 8 | $\angle ECB + \angle ECD =$ r$\angle$. |
| ∵ e. 8. 6. | 9 | $\angle ECB + \angle ECD = \angle EDC + \angle ECD$. *Ax.* 1. |
| 9—$\angle ECD$ | 10 | $\angle ECB = \angle EDC$. *Ax.* 12. |
| Laſtly | 11 | $\angle EDC + \angle EFC =$ 2 r$\angle$'s. 22. |
| ∵ e. 10. 11. | 12 | $\angle ECB + \angle EFC =$ 2 r$\angle$'s. *Sub.* |
| Alſo | 13 | $\angle ECB + \angle EGA =$ 2 r$\angle$'s. 13. 1. |
| ∵ e. 13. 12. | 14 | $\angle ECB + \angle ECA = \angle ECB + \angle EFC$. *Ax.* 1. |
| 14—$\angle ECB$ | 15 | $\angle ECA = \angle EFC$. *Ax.* 12. |
| ∵ e. 10. 15. | 16 | $\angle ECB = \angle EDC$. $\angle ECA = \angle EFC$. Q. E. D. |

## PROP. XXXIII.

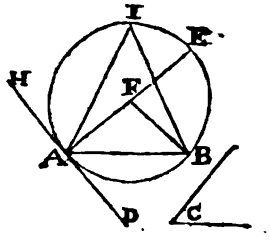
*Upon a Right-line* A B, *to deſcribe a Segment of a Circle* A I E B, *which ſhall contain an Angle* A I B, *equal to a given Right-lined Angle* C.

### PREPARATION.

Make (23. 1.) ∠ B A D = ∠ C. Through the Point A, draw (11. 1.) the Line A E perpendicular to H D. At one End of the Line A B, make ∠ A B F = ∠ B A F, and let one ſide thereof cut A E in F. From the Center F, thro' the Point A, deſcribe a Circle, which (becauſe ∠ A B F = ∠ B A F, and therefore (6. 1.) F B = F A) will alſo paſs thro' B; and to any Point I, in the Periphery A I E B, draw A I, B I.

I ſay, ∠ A I B = ∠ C.

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | H D is drawn through the Extremity of the Diameter A E at r ∠ᵉ. |
| ∵ e. 1. | 2 | H D touches the Circle. *Cor.* 16. |
| Again | 3 | A B drawn from the Point of Contact, cuts the Circle. |
| ∵ e. 2. 3. | 4 | H D touches, and A B drawn from the Point of Contact, cuts the Circle. |
| ∵ e. 4. | 5 | ∠ A I B = ∠ B A D. 32. |
| Alſo | 6 | ∠ C = ∠ B A D. *Conſt.* |
| ∵ e. 5. 6. | 7 | ∠ A I B = ∠ C. *Ax.* 1. Q. E. F. & D. |

## PROP. XXXIV.

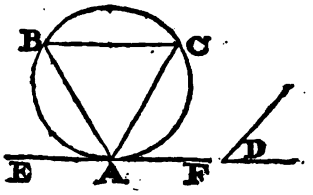
*From a Circle given* A B C, *to cut off a Segment* A B C, *containing an Angle* B, *equal to a Right-lined Angle given* D.

PRE-

## PREPARATION.

Draw (17.) a Right-line E F, touching the Circle given in A; alſo draw A C, making an $\angle F A C = \angle D$; and to any Poiut B, in the Periphery A B C, draw A B, C B.

I ſay, $\angle \mathfrak{B} = \angle \mathfrak{D}$.




## DEMONSTRATION.

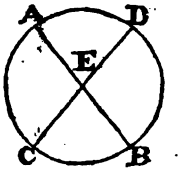
| | | |
|---|---|---|
| | 1 | E F touches, and A C drawn from the point of Contact, cuts the Circle. |
| ∴ e. 1. | 2 | $\angle F A C = \angle B$. 32. |
| Alſo | 3 | $\angle F A C = \angle D$. *Conſt.* |
| ∴ e. 2. 3. | 4 | $\angle \mathfrak{B} = \angle \mathfrak{D}$. *Ax.* 1. Q. E. F. & D. |

## PROP. XXXV.

*If in a Circle* A C B D, *two Right-lines* A B, D C, *cut each other, the Rectangle contained under the Segments* A E, E B, *of the one, is equal to the Rectangle contained under the Segments* C E, E D, *of the other.*

*Caſe* 1 If the two Lines cut each other in the Center E.

I ſay, $\mathfrak{A}\mathfrak{E} \times \mathfrak{E}\mathfrak{B} = \mathfrak{C}\mathfrak{E} \times \mathfrak{E}\mathfrak{D}$.




## DEMONSTRATION.

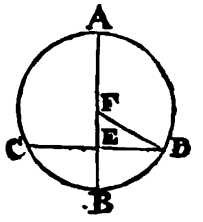
| | | |
|---|---|---|
| | 1 | A E = C E. *Def.* 16. 1. |
| And | 2 | E B = E D. *Def.* 16. 1. |
| 1 × 2 | 3 | $\mathfrak{A}\mathfrak{E} \times \mathfrak{E}\mathfrak{B} = \mathfrak{C}\mathfrak{E} \times \mathfrak{E}\mathfrak{D}$. *Ax.* 13. Q. E. D. |

*Case* 2 If one Line A B, pafs through the Center F, and cut the other D C equally, and conſequently (3.) at Right-Angles) in E.

## PREPARATION.

Draw F D.

I ſay, $AE \times EB = CE \times ED$.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $AE \times EB + EF^q = FB^q$. 5. 2. |
| And | 2 | $FD = FB$. *Def.* 16. 1. |
| 2 & 2. | 3 | $FD^q = FB^q$. *Ax.* 9. |
| ∴ e. 1. 3. | 4 | $AE \times EB + EF^q = FD^q$. *Ax.* 1. |
| Alſo | 5 | $ED^q + EF^q = FD^q$. 47. 1. |
| ∴ e. 4. 5. | 6 | $AE \times EB + EF^q = ED^q + EF^q$. *Ax.* 1. |
| 6 — $EF^q$. | 7 | $AE \times EB = ED^q$. *Ax.* 12. |
| Again | 8 | $CE = ED$. *Conſt.* |
| And | 9 | $CE \times ED = CE \times ED$. |
| ∴ e. 8. 9. | 10 | $CE \times ED = ED \times ED$. *Sub.* |
| i. e. | 11 | $CE \times ED = ED^q$. |
| ∴ e. 7. 11. | 12 | $AE \times EB = CE \times ED$. *Ax.* 1. Q. E. D. |

*Case* 3 If one Line AB, pafs through the Center F, and cut the other D C unequally in E.

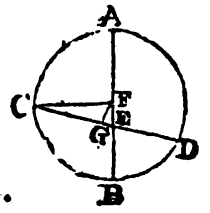
## PREPARATION.

Biſect (3.) D C, by F G a Perpendicular from the Center; and draw F C.

I ſay, $AE \times EB = CE \times ED$.

D E-

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $CE \times ED + EG^q = CG^q$. 5. 2. |
| $1 + FG^q$ | 2 | $CE \times ED + EG^q + FG^q = CG^q + FG^q$. *Ax.* 11. |
| Again | 3 | $EG^q + FG^q = FE^q$. 47. 1. |
| ∴ e. 2. 3. | 4 | $CE \times ED + FE^q = CG^q + FG^q$. *Sub.* |
| Alſo | 5 | $FC^q = CG^q + FG^q$. 47. 1. |
| ∴ e. 4. 5. | 6 | $CE \times ED + FE^q = FC^q$. *Ax.* 1. |
| Laſtly | 7 | $FB = FC$. *Def.* 16. 1. |
| 7 ⊙ 2. | 8 | $FB^q = FC^q$. *Ax.* 9. |
| ∴ e. 6. 8. | 9 | $CE \times ED + FE^q = FB^q$. *Ax.* 1. |
| Alſo | 10 | $AE \times EB + FE^q = FB^q$. 5. 2. |
| ∴ e. 10. 9. | 11 | $AE \times EB + FE^q = CE \times ED + FE^q$. *Ax.* 1. |
| 11 — $FE^q$. | 12 | $AE \times EB = CE \times ED$. *Ax.* 12. Q. E. D. |

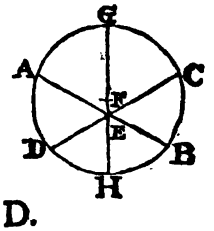
*Caſe.* 4. If neither of the two Lines AB, DC, paſs thro' the Center F.

## PREPARATION.

Thro' the Point of Interſection E, draw the Diameter GH.

I ſay, $AE \times EB = CE \times ED$.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $AE \times EB = GE \times EH$. *Caſe* 3. |
| Alſo | 2 | $CE \times ED = GE \times EH$. *Caſe* 3. |
| ∴ e. 1. 2. | 3 | $AE \times EB = CE \times ED$. *Ax.* 1. Q. E. D. |

## PROP. XXXVI.

*A Tangent* DB, *and Secant* DA, *being drawn from the ſame Point* D, *taken at Pleaſure without the Circle* ABC; *the Square of the Tangent* DB, *will be equal to the Rectangle under the whole Secant* DA, *and its external Part* DC.

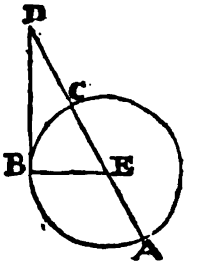
*Caſe* 1 If the Secant DA paſs thro' the Center E.

### PREPARATION.

From the Center E, to the Point of Contact B, draw EB, which (18.) will be Perpendicular to the Tangent DB.

I ſay, $\square DB = DA \times DC$.

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $EB = EC$. *Def.* 16. 1. |
| 1 & 2. | 2 | $EB^q = EC^q$. *Ax.* 9. |
| Again | 3 | $DB^q + EB^q = ED^q$. 47. 1. |
| ∵ 2. 3. | 4 | $DB^q + EC^q = ED^q$. *Sub.* |
| Alſo | 5 | $DA \times DC + EC^q = ED^q$. 6. 2. |
| ∵ 4. 5. | 6 | $DB^q + EC^q = DA \times DC + EC^q$. *Ax.* 1. |
| 6 — $EC^q$. | 7 | $\square DB = DA \times DC$. *Ax.* 13. Q. E. D. |

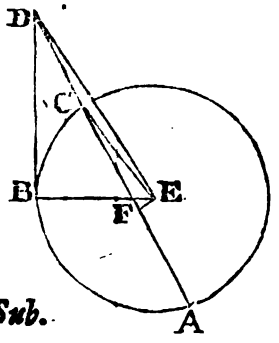
*Caſe* 2 If the Secant DA paſs not thro' the Center E.

### PREPARATION.

From the Center E, to the Point of Contact B, draw EB, which (18.) will be perpendicular to the Tangent DB; alſo draw EC. ED, and EF cutting DA, at Right-angles, which (3.) will Biſect CA in the Point F.

I ſay, $\square DB = DA \times DC$. DEMON-

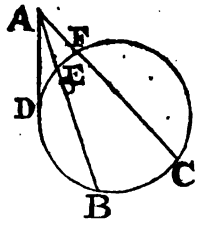
## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $DB^q + EB^q = ED^q$. 47. 1. |
| Alſo | 2 | $DF^q + EF^q = ED^q$. 47. 1. |
| ∵ e. 1. 2. | 3 | $DB^q + EB^q = DF^q + EF^q$. *Ax.* 1. |
| Again | 4 | $DA \times DC + CF^q = DF^q$. 6. 2. |
| ∵ e. 3. 4. | 5 | $DB^q + EB^q = DA \times DC + CF^q + EF^q$. *Sub.* |
| And | 6 | $CF^q + EF^q = EC^q$. 47. 1. |
| ∵ e. 5. 6. | 7 | $DB^q + EB^q = DA \times DC + EC^q$. *Sub.* |
| Laſtly | 8 | $EB = EC$. *Def.* 16. 1. |
| 8 & 2. | 9 | $EB^q = EC^q$. *Ax.* 9. |
| ∵ e. 7. 9. | 10 | $DB^q + EC^q = DA \times DC + EC^q$. *Sub.* |
| 10—$EC^q$. | 11 | $DB^q = DA \times DC$. *Ax.* 12. Q. E. D. |

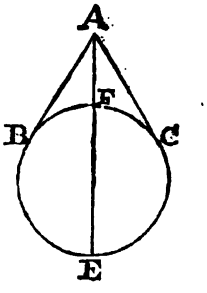
## COROLLARIES.

1. If from any Point A, taken without a Circle, there be drawn ſeveral Right-lines AB, AC, cutting the Circle, the Rectangles contained under the whole Lines AB, AC, and the external Parts AE, AF, are equal between themſelves.




| | | |
|---|---|---|
| For | 1 | If the Tangent AD be drawn. |
| Then | 2 | $AB \times AE = AD^q$. 36. |
| Alſo | 3 | $AC \times AF = AD^q$. 36. |
| ∵ e. 2. 3. | 4 | $AB \times AE = AC \times AF$. *Ax.* 1. |

2. If two Lines AB, AC, drawn from the ſame Point A, touch the Circle, thoſe two Lines are equal one to the other.




| | | |
|---|---|---|
| For | 1 | If the Secant AE be drawn. |
| Then | 2 | $AE \times AF = AB^q$. 36. |
| Alſo | 3 | $AE \times AF = AC^q$. 36. |
| ∵ e. 2. 3. | 4 | $AB^q = AC^q$. *Ax.* 1. |
| 4 *in* 2. | 5 | $AB = AC$. *Ax.* 10. |

## PROP. XXXVII.

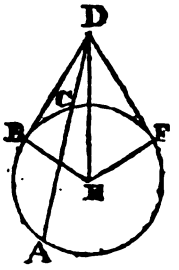
*If the Rectangle contained under the Secant* D A, *and its external Part* D C, *be equal to the Square of a Line* D B, *meeting the Circumference of a Circle* A B C F, *that Line* D B *touches the Circle.*

### PREPARATION.

From the Point D, draw (17.) a Tangent D F; alſo from the Center E, draw E B, E D, E F.

I ſay, D B touches the Circle.

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | D A × D C = D F *q*. 36. |
| Alſo | 2 | D A × D C = D B *q*. *Hyp.* |
| ∵ c. 1. 2. | 3 | D F *q* = D B *q*. *Ax.* 1. |
| 3 *to* 2. | 4 | D F = D B. *Ax.* 10. |
| Again | 5 | E F = E B. *Def.* 16. 1. |
| And | 6 | E D is common. |
| ∵ c. 4. 5. 6. | 7 | D F = D B, E F = E B, E D is common. |
| ∵ c. 7. | 8 | ∠ E F D = ∠ E B D. 8. 1. |
| But | 9 | E F D is a r∠. 18. |
| ∵ c. 8. 9. | 10 | E B D is a r∠. |
| ∵ c. 10. | 11 | D B touches the Circle. *Cor.* 16. Q. E. D. |

*The End of the Third Book.*

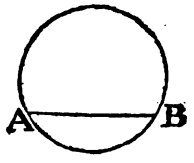
# EUCLID's ELEMENTS,

Demonſtrated after a New, Plain, and Eaſie METHOD.

## BOOK IV.

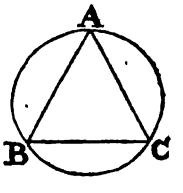
### DEFINITIONS.

1 A **Right-line is ſaid to be applied in a Circle,** when the Extreams thereof fall upon the Circumference.




As the Line AB.

2 **A Right-lined Figure is ſaid to be inſcribed in a Circle,** when the Vertex of each of its Angles touches the Circumference of the Circle it is inſcribed in.

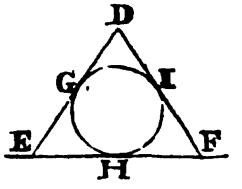



Thus the Triangle ABC is inſcribed in the Circle ABC.

3 **A Circle is said to be described about a Right-lined Figure,** when its Circumference passes thro' the Vertex of each Angle of the Figure it is said to be circumscribed about.

Thus the Circle ABC is described about the Triangle ABC.

4 **A Right-lined Figure is said to be described about a Circle,** when all the Sides of the Figure which is circumscribed, touch the Periphery of the Circle.




Thus the Triangle EDF, is described about the Circle GHI.

5 **A Circle is said to be inscribed in a Right-lined Figure,** when the Periphery of the Circle touches all the Sides of the Figure in which it is inscribed.

Thus the Circle GHI, is inscribed in the Triangle EDF.

6 **Regular Pentagons, Heragons,** and **Pentedecagons,** are such as have equal Sides, and equal Angles.

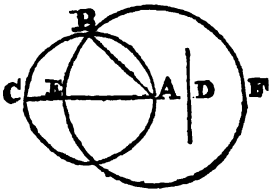
## PROP. I.

*In a Circle given* ABC, *to apply a Right-line* AB, *equal to a Right-line given* D, *which doth not exceed* AC *the Diameter of the Circle.*

## PREPARATION.

From the Center A, with the Distance AE = D, describe a Circle BEF, meeting with the Circle given in B, and draw AB.

I ſay, 𝔄𝔅 = 𝔇.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AE = AB. *Def.* 16. 1. |
| Alſo | 2 | AE = D. *Conſt.* |
| ∵ e. 1. 2. | 3 | 𝔄𝔅 = 𝔇. *Ax.* 1. Q. E. F. & D. |

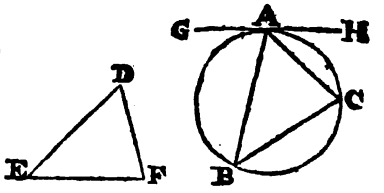
## PROP. II.

*In a Circle given* ABC, *to inſcribe a Triangle* ABC, *Equiangular to a Triangle given* DEF.

## PREPARATION.

Draw (17. 3.) the Tangent GH, and making (23. 1.) ∠HAC = ∠E, ∠GAB = ∠F, join BC; ſo (*Def.* 2.) is the Triangle ABC inſcribed in the Circle ABC.

I ſay, ∠𝔅 = ∠𝔈. ∠𝔈 = ∠𝔉. ∠𝔈𝔄𝔅 = ∠𝔇.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | ∠HAC = ∠B. 32. 3. |
| Alſo | 2 | ∠HAC = ∠E. *Conſt.* |
| ∵ e. 1. 2. | 3 | ∠B = ∠E. *Ax.* 1. |
| Again | 4 | ∠GAB = ∠C. 32. 3. |
| Alſo | 5 | ∠GAB = ∠F. *Conſt.* |
| ∵ e. 4. 5. | 6 | ∠C = ∠F. *Ax.* 1. |
| ∵ e. 3. 6. | 7 | ∠B = ∠E, ∠C = ∠F. |
| ∵ e. 7. | 8 | ∠CAB = ∠D. *Cor.* 2. 32. 1. |
| ∵ e. 7. 8. | 9 | ∠𝔅 = ∠𝔈. ∠𝔈 = ∠𝔉. ∠𝔈𝔄𝔅 = ∠𝔇. Q. E. F. & D. |

## PROP. III.

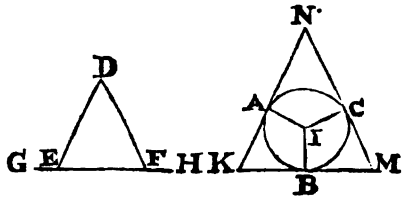
*About a Circle given* ABC, *to describe a Triangle* KNM, *Equiangular to a Triangle given* DEF.

## PREPARATION.

Produce the Side EF on both Sides; at the Center I, make (23. 1.) ∠AIB = ∠DEG, ∠BIC = ∠DFH. Thro' the Points A, B, C, draw (17. 3.) three Tangents meeting in K, N, M, so (*Def.* 4.) is the Triangle KNM, described about the Circle ABC; also the Radij IA, IB, IC, are (18. 3.) perpendicular to KN, KM, MN, and consequently the Angles at A, B, C, are (*Def.* 10. 1.) Right.

I say, ∠𝔎 = ∠𝔇𝔈𝔉. ∠𝔐 = ∠𝔇𝔉𝔈. ∠𝔑 = ∠𝔇.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | ∠A + ∠B + ∠AIB + ∠K = 4r∠s. *Cor.* 2. *The.* 1. 1. |
| And | 2 | ∠A + ∠B = 2r∠s. *Const.* |
| 1 — 2 | 3 | ∠AIB + ∠K = 2r∠s. *Ax.* 12. |
| Also | 4 | ∠DEG + ∠DEF = 2r∠s. 13. 1. |
| ∴ e. 3. 4. | 5 | ∠AIB + ∠K = ∠DEG + ∠DEF. *Ax.* 1. |
| But | 6 | ∠AIB = ∠DEG. *Const.* |
| ∴ e. 5. 6. | 7 | ∠AIB + ∠K = ∠AIB + ∠DEF. *Sub.* |
| 7 — ∠AIB. | 8 | ∠K = ∠DEF. *Ax.* 12. |
| Again | 9 | ∠B + ∠C + ∠BIC + ∠M = 4r∠s. *Cor.* 2. *The.* 1. 1. |
| And | 10 | ∠B + ∠C = 2r∠s. *Const.* |

∠BIC

| | | |
|---|---|---|
| 9—10. | 11 | $\angle BIC + \angle M = 2r\angle$'. *Ax.* 12. |
| Alſo | 12 | $\angle DFH + \angle DFE = 2r\angle$'. 13. 1. |
| ∴e. 11. 12. | 13 | $\angle BIC + \angle M = \angle DFH + \angle DFE$. *Ax.* 1. |
| Laſtly | 14 | $\angle BIC = \angle DFH$. *Conſt.* |
| ∴e. 13. 14. | 15 | $\angle BIC + \angle M = \angle BIC + \angle DFE$. *Sub.* |
| 15—∠BIC. | 16 | $\angle M = \angle DFE$. *Ax.* 12. |
| ∴ e. 8. 16. | 17 | $\angle K = \angle DEF$, $\angle M = \angle DFE$. |
| ∴e. 17. | 18 | $\angle N = \angle D$. *Cor.* 2. 32. 1. |
| ∴e. 17. 18. | 19 | $\angle \mathfrak{K} = \angle \mathfrak{D}\,\mathfrak{E}\,\mathfrak{F}$. $\angle \mathfrak{M} = \angle \mathfrak{D}\,\mathfrak{F}\,\mathfrak{E}$. $\angle \mathfrak{N} = \angle \mathfrak{D}$. Q. E. F. & D. |

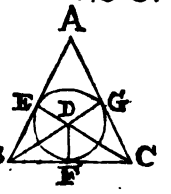
## PROP. IV.

*To inſcribe a Circle* EFG, *in a given Triangle* ABC.

### PREPARATION.

Biſect (9. 1.) the Angles B, C, with the Right-lines BD, CD, meeting in the Point D; then having (12. 1.) drawn the Perpendiculars DE, DF, DG, ſo that DEB, DFB, DFC, DGC, may (*Def.* 10. 1.) be Right-angles, from the Center D, with either of theſe Diſtances, deſcribe the Circle EFG.

I ſay, The Circle EFG is inſcribed in the △ ABC.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\angle DEB = \angle DFB$. *Ax.* 21. |
| And | 2 | $\angle DBE = \angle DBF$, DB is common. *Conſt.* |
| ∴e. 1. 2. | 3 | $\angle DEB = \angle DFB$, $\angle DBE = \angle DBF$, DB is com. |
| ∴e. 3. | 4 | DE = DF. 26. 1. |
| Again | 5 | $\angle DFC = \angle DGC$. *Ax.* 21. |
| And | 6 | $\angle DCF = \angle DCG$, DC is common. *Conſt.* |
| ∴e. 5. 6. | 7 | $\angle DFC = \angle DGC$, $\angle DCF = \angle DCG$, DC is common. |
| ∴e. 7. | 8 | DF = DG. 26. 1. |
| ∴e. 4. 8 | 9 | DE = DF = DG. *Ax.* 1. |
| ∴e. 9. | 10 | The Circle EFG paſſes thro' the Points E, F, G. *Def.* 16. 1. |
| Alſo | 11 | DEB, DFB, DGC, are $r\angle$'. *Conſt.* |
| ∴e. 11. | 12 | The Periphery of the Circle EFG touches all the Sides of the △ ABC. *Cor.* 16. 3. |
| ∴e. 12. | 13 | The Circle EFG is inſcribed in the △ ABC. *Def.* 5. Q. E. F. & D. |

## PROP. V.

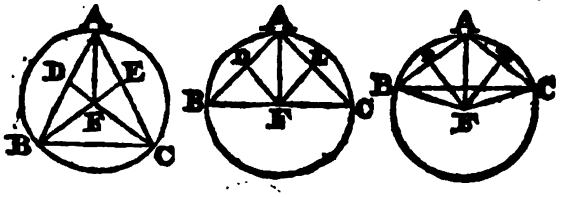
*About a given Triangle* ABC, *to describe a Circle* ABC.

## PREPARATION.

Biſect (10. 1) any two Sides AB, AC, in the Points D, E, and having (11. 1.) drawn from thoſe Points D, E, the Lines DF, EF, at Right-angles to the Sides AB, AC, (which will meet, either within the Triangle ABC, or in the Line BC, or without in the Point F,) draw FA, FB, FC, and from the Point F, with either of thoſe Diſtances, deſcribe the Circle ABC.

I ſay, **The Circle ABC is deſcribed about the △ABC.**

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AD = DB, DF is common. *Conſt.* |
| And | 2 | ∠FDA = FDB. *Ax.* 21. |
| ∴ e. 1. 2. | 3 | AD = DB, DF is common, ∠FDA = ∠FDB. |
| ∴ e. 3. | 4 | FA = FB. 4. 1. |
| Again | 5 | AE = EC, FE is common. *Conſt.* |
| And | 6 | ∠FEA = ∠FEC. *Ax.* 21. |
| ∴ e. 5. 6. | 7 | AE = EC, FE is common, ∠FEA = ∠FEC. |
| ∴ e. 7. | 8 | FA = FC. 4. 1. |
| ∴ e. 4. 8. | 9 | FA = FB = FC. *Ax.* 1. |
| ∴ e. 9. | 10 | The Circumference of the Circle ABC, paſſes thro' the Vertex of each of the ∠ˢ A, B, C. *Def.* 16. 1. |
| ∴ e 10. | 11 | **The Circle ABC is deſcribed about the △ABC.** *Def.* 1. Q. E. F. & D. |

SCH

## SCHOLIUM.

By the same Method, may a Circle be described, that shall pass thro' three Points given, not being in the same straight Line.

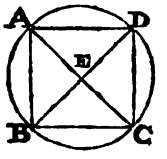
## PROP. VI.

*In a Circle given* ABCD, *to inscribe a Square* ABCD.

### PREPARATION.

Draw (11. 1.) the Diameters AC, DB, cutting each other at Right-angles in the Center E; then join their Extremes A, B, C, D, with the Right-lines AB, BC, CD, DA.

I say, **The Square ABCD is inscribed in the Circle ABCD.**




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AEB, BEC, CED, DEA, are r∠'. *Const.* |
| ∴ e. 1. | 2 | ∠AEB = ∠BEC = ∠CED = ∠DEA. *Ax.* 21. |
| ∴ e. 2. | 3 | Arch AB = Arch BC = Arch CD = Arch DA. 26. 3. |
| ∴ e. 3. | 4 | AB = BC = CD = DA. 29. 3. |
| Again | 5 | ABC, BCD, CDA, DAB, are Semicircles. *Def.* 18. 1. |
| ∴ e. 5. | 6 | ABC, BCD, CDA, DAB, are r∠'. 31. 3. |
| ∴ e. 6. | 7 | ∠ABC = ∠BCD = ∠CDA = ∠DAB. *Ax.* 21. |
| ∴ e. 4. 7. | 8 | AB = BC = CD = DA; ∠ABC = ∠BCD = ∠CDA = ∠DAB. |
| ∴ e. 8. | 9 | ABCD is a Square. *Def.* 29. 1. |
| Also | 10 | The Vertex of each of its Angles touches the Circumference of the Circle ABCD. |
| ∴ e. 10. | 11 | **The Square ABCD is inscribed in the Circle ABCD.** *Def.* 2. Q. E. F. & D. |

## PROP. VII.

*About a Circle given* ABCD, *to deſcribe a Square* FHIG.

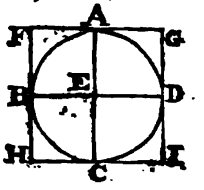
## PREPARATION.

Draw (11. 1.) the Diameters AC, DB, cutting each other at Right-angles in the Center E; then thro' their Extremes A, B, C, D, draw (17. 3.) Tangents meeting in F, H, I, G, ſo (18. 3.) will EA, EB, EC, ED be perpendicular to thoſe Tangents, and therefore (*Def.* 10. 1.) will the Angles at A, B, C, D, be Right.

I ſay, **The Square FHIG is deſcribed about the Circle ABCD.**

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | The ∠'s at A, B, C, D, E, are Right. *Conſt.* |
| ∵ e. 1. | 2 | FG ‖ HI, FH ‖ GI; FA ‖ HC, FH ‖ AC; FB ‖ GD, FG ‖ BD. 28. 1 |
| ∵ e. 2. | 3 | FHIG, FHCA, FBDG, are □'s. *Def.* 35. 1. |
| ∵ e. 3. | 4 | FH = AC, BD = FG. 34. 1. |
| Alſo | 5 | AC = BD. *Def.* 16. 1. |
| ∵ e. 4. 5. | 6 | FH = AC, AC = FG. *Sub.* |
| ∵ e. 6. | 7 | FH = FG. *Ax.* 1. |
| Alſo e. 3. | 8 | HI = FG, FH = GI. 34. 1. |
| ∵ e. 7. 8. | 9 | HI = FG = FH = GI. *Ax.* 1. |
| Again | 10 | ∠B + ∠E + ∠A + ∠F = 4r∠'s. ∠A + ∠E + ∠D + ∠G = 4r∠'s. *Cor.* 2. *Theo.* 1. 1. |
| Alſo | 11 | ∠B + ∠E + ∠A = 3r∠'s. ∠A + ∠E + ∠D = 3r∠'s. *Conſt.* |
| ∵ e. 10. 11. | 12 | ∠F = r∠, ∠G = r∠. |
| ∵ e. 12. | 13 | ∠F = ∠G. *Ax.* 21. |
| Laſtly | 14 | ∠I = ∠F; ∠G = ∠H. 34. 1. |
| ∵ e. 13. 14. | 15 | ∠I = ∠F = ∠G = ∠H. *Ax.* 1. |
| ∵ e. 9. 15. | 16 | HI = FG = FH = GI; ∠I = ∠F = ∠G = ∠H. |
| ∵ e 16 | 17 | FHIG is a Square. *Def.* 29. 1. |
| Alſo | 18 | All its Sides touch the Periphery of the Circle ABCD. *Conſt.* |
| ∵ e. 18. | 19 | **The Square FHIG is deſcribed about the Circle ABCD.** *Def.* 4. Q. E. F. & D. |




PROP.

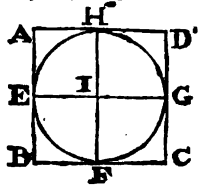
## PROP. VIII.

*In a Square given* ABCD, *to inſcribe a Circle* HEFG.

## PREPARATION.

Biſect (10. 1.) the oppoſite Sides of the Square ABCD, in the Points E, G, H, F, which being *(Def.* 29 1.*)* equal, as alſo *(*28. 1.*)* parallel, their Halves AE, DG, EB, GC, AH, BF, HD, FC, will *(Ax.* 8.*)* be equal, as alſo parallel; then having drawn EG, HF, from the Point of Interſection I, at the Diſtance IH, deſcribe the Circle HEFG.

I ſay, **The Circle HEFG is inſcribed in the Square ABCD.**




## DEMONSTRATION.

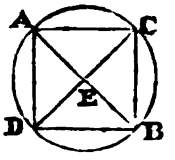
| | | |
|---|---|---|
| | 1 | AE ‖ and = DG; EB ‖ and = GC. *Conſt.* |
| ∵ e. 1. | 2 | AD ‖ EG; EG ‖ BC. 33. 1. |
| Alſo | 3 | AH ‖ and = BF; HD ‖ and = FC. *Conſt.* |
| ∵ e. 3. | 4 | AB ‖ HF; HF ‖ DC. 33. 1. |
| ∵ e. 2. 4. | 5 | AD ‖ EG, EG ‖ BC, AB ‖ HF, HF ‖ DC. |
| ∵ e. 5. | 6 | IA, IB, IC, are □ˢ. *Def.* 35. 1. |
| ∵ e. 6. | 7 | IH=AE, AH=IE, IE=BF, EB=IF, IF=GC, FC=IG. 34. 1. |
| Again | 8 | AE=AH, BF=EB, GC=FC. *Conſt.* |
| ∵ e. 7. 8. | 9 | IH=AE, AE=IE; IE=BF, BF=IF; IF=GC, GC=IG. *Sub.* |
| ∵ e. 9. | 10 | IH = IE = IF = IG. *Ax.* 1. |
| ∵ e. 10. | 11 | The Circle HEFG paſſes thro' the Points H, E, F, G. *Def.* 16. 1. |
| Laſtly e. 6. | 12 | IH ‖ AE, IE ‖ AH, IF ‖ GC, IG ‖ FC. *Def.* 35. 1. |
| ∵ e. 12. | 13 | ∠A + ∠H = 2r∠ˢ, ∠A + ∠E = 2r∠ˢ, ∠C + ∠F = 2r∠ˢ, ∠C + ∠G = 2r∠ˢ. 29. 1. |
| But | 14 | ∠A = r∠, ∠C = r∠. *Def.* 29. 1. |
| ∵ e. 13. 14. | 15 | H, E, F, G, are r∠ˢ. |
| ∵ e. 15. | 16 | The Periphery of the Circle HEFG touches all the Sides of the Square ABCD. *Cor.* 16. 3. |
| ∵ e. 16. | 17 | **The Circle HEFG is inſcribed in the Square ABCD.** *Def.* 5. Q. E. F. & D. |

## PROP. IX.

*About a given Square* ADBC, *to deſcribe a Circle* ADBC.

### PREPARATION.

Draw the Diagonals AB, CD, cutting each other in the Point E; then from the Center E, at the Diſtance EA, deſcribe the Circle ADBC.




I ſay, **The Circle ADBC is deſcribed about the Square ADBC.**

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CA = CB, ∠ACB is Right; DA = DB, ∠ADB is Right. *Def.* 29. 1. |
| ∵ e. 1. | 2 | ∠CAB, DAB, DBA, are Semiright. *Cor.* 6. 32. 1. |
| Alſo | 3 | AC = AD, ∠DAC is Right; BC = BD, ∠DBC is Right. *Def.* 29. 1. |
| ∵ e 3. | 4 | ∠ACD, ADC, BDC, are Semiright. *Cor.* 6. 32. 1. |
| ∵ e. 2. 4 | 5 | ∠CAB = ∠ACD, ∠DAB = ∠ADC, ∠DBA = ∠BDC. |
| i. e. | 6 | ∠CAE = ∠ACE, ∠DAE = ∠ADE, ∠DBE = ∠BDE. |
| ∵ e. 6. | 7 | EC = EA, EA = ED, ED = EB. 6. 1. |
| ∵ e. 7. | 8 | EC = EA = ED = EB. *Ax.* 1. |
| ∵ e. 8. | 9 | The Circumference of the Circle ADBC, paſſes thro' the Vertex of each of the ∠'s A, D, B, C. *Def.* 16. 1. |
| ∵ e. 9. | 10 | **The Circle ADBC is deſcribed about the Square ADBC.** *Def.* 3. Q. E. F. & D. |

## PROP. X.

*To make an Iſoſceles Triangle* ABC, *having each Angle at the Baſe,* A, BCA, *double to the remaining Angle* B.

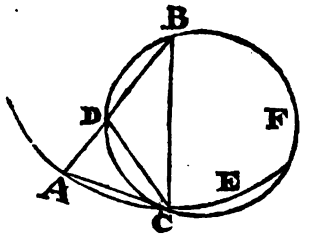
### PREPARATION.

Draw the Line BA at pleaſure, and (11. 2.) divide it at the Point D, ſo that the Square of BD, be equal to the Rectangle under BA, AD; then from the Center B, at the Diſtance BA, deſcribe the Arch ACE, in which (1.) ap-

(1.) apply AC = BD; and drawing the Lines BC, CD, about the Triangle BDC, describe (5.) a Circle BDCF.

I say, BA = BC. ∠A = ∠BCA = 2∠B.

DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AC = BD. *Const.* |
| 1 e. 2. | 2 | AC$q$ = BD$q$. *Ax.* 9. |
| Also | 3 | BA × AD = BD$q$. *Const.* |
| ∵ e. 3. 2. | 4 | BA × AD = AC$q$. *Ax.* 1. |
| ∵ e. 4. | 5 | AC touches the Circle. 37. 3. |
| Also | 6 | CD drawn from the Point of Contact cuts the Circle. |
| ∵ e. 5. 6. | 7 | AC touches, and CD drawn from the Point of Contact cuts the Circle. |
| ∵ e 7. | 8 | ∠ACD = ∠B. 32. 3. |
| Again | 9 | ∠ADC = ∠B + ∠BCD. 32. 1. |
| ∵ e. 8. 9. | 10 | ∠ADC = ∠ACD + ∠BCD. *Sub.* |
| i. e. | 11 | ∠ADC = ∠BCA. |
| Moreover | 12 | BA = BC. *Def.* 16. 1. |
| ∵ e. 12. | 13 | ∠A = ∠BCA. 5. 1. |
| ∵ e. 11. 13. | 14 | ∠ADC = ∠A. *Ax.* 1. |
| ∵ e. 14. | 15 | AC = CD. 6. 1. |
| ∵ e. 1. 15. | 16 | BD = CD. *Ax.* 1. |
| ∵ e. 16. | 17 | ∠BCD = ∠B. 5. 1. |
| ∵ e. 8. 17. | 18 | ∠ACD = ∠BCD. *Ax.* 1. |
| Now | 19 | ∠BCD + ∠ACD = ∠BCA. |
| ∵ e. 18. 19. | 20 | ∠ACD + ∠ACD = ∠BCA. *Sub.* |
| i. e. | 21 | 2∠ACD = ∠BCA. |
| 8 × 2 | 22 | 2∠ACD = 2∠B. *Ax.* 13. |
| ∵ e. 21. 22. | 23 | ∠BCA = 2∠B. *Ax.* 1. |
| ∵ e. 13. 23. | 24 | ∠A = ∠BCA = 2∠B. *Ax.* 1. |
| ∵ e. 12. 24. | 25 | BA = BC. ∠A = ∠BCA = 2∠B. Q. E. F. & D. |

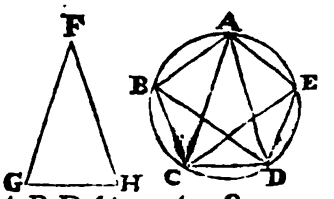
## PROP. XI.

*In a Circle given* ABCDE, *to inſcrile a regular Pentagon* ABCDE.

### PREPARATION.

Deſcribe (10.) an Iſoſceles Triangle FGH, having each Angle at the Baſe double to the remaining Angle, and (2.) inſcribe in the given Circle, the Triangle ACD Equiangular to the Triangle FGH; then (9. 1.) biſect the Angles at the Baſe ACD, ADC, with the Right-lines CE, DB, meeting with the Circumference in E and B, and join CB, BA, AE, ED.

I ſay, **The regular Pentagon ABCDE is inſcribed in the Circle ABCDE.**




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\triangle ACD = 2\angle CAD = \angle ADC$. *Conſt.* |
| ∵ e. 1. | 2 | $\angle ACD = \angle ADC$ *Ax.* 7. |
| ∵ e. 2. | 3 | $\angle ACE = \angle ECD = \angle ADB = \angle BDC$. *Ax.* 8. |
| Again | 4 | $\angle BDC = \angle CAD$. *Conſt.* |
| ∵ e. 3. 4. | 5 | $\angle ACE = \angle ECD = \angle ADB = \angle BDC = \angle CAD$. *Ax.* 1. |
| ∵ e. 5. | 6 | Arch AE = Arch ED = Arch AB = Arch BC = Arch CD. 26. 3. |
| ∵ e. 6. | 7 | AE = ED = AB = BC = CD. 29. 3. |
| Laſtly | 8 | Each of the $\angle$'s ABC, BCD, CDE, DEA, EAB, ſtand on three of the equal Arches. |
| ∵ e. 8. | 9 | $\angle ABC = \angle BCD = \angle CDE = \angle DEA = \angle EAB$. 27. 3. |
| ∵ e. 7. 9. | 10 | AE = ED = AB = BC = CD; $\angle ABC = \angle BCD = \angle CDE = \angle DEA = \angle EAB$. |
| ∵ e. 10. | 11 | ABCDE is a regular Pentagon. *Def.* 6. |
| Alſo | 12 | The Vertex of each of its Angles touches the Circumference of the Circle ABCDE. |
| ∵ e. 12 | 13 | **The regular Pentagon ABCDE is inſcribed in the Circle ABCDE.** *Def.* 2. Q. E. F. & D. |

PROP.

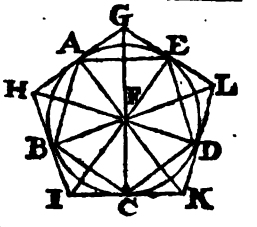
## PROP. XII.

*About a Circle given* ABCDE, *to describe a regular Pentagon* GHIKL.

### PREPARATION.

Inscribe (11.) a regular Pentagon ABCDE in the Circle given; then having from the Center F, drawn the Right-lines FA, FB, FC, FD, FE, draw to those Lines so many Perpendiculars GAH, HBI, ICK, KDL, LEG, so (*Def.* 10. 1.) will all the Angles at A, B, C, D, E, be Right, and consequently (*Ax.* 21.) equal. Lastly, draw FG, FH, FI, FK, FL.

I say, **The regular Pentagon GHIKL is described about the Circle ABCDE.**




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | GA, GE, touch the Circle. *Cor.* 16. 3. |
| ∴ e. 1. | 2 | GA = GE. *Cor.* 2. 36. 3. |
| Again | 3 | FA = FE. *Def.* 16. 1. |
| And | 4 | FG is common. |
| ∴ e. 2. 3. 4. | 5 | GA = GE, FA = FE, FG is common. |
| ∴ e. 5. | 6 | ∠GFA = ∠GFE. 8. 1. |
| ∴ e. 6. | 7 | ∠AFE is bisected. |
| Also | 8 | After the same manner, are the ∠'AFB, BFC, CFD, DFE, bisected. |
| Now | 9 | ∠AFE = ∠AFB = ∠BFC = ∠CFD = ∠DFE. 27. 3. |
| ∴ e. 9. | 10 | All the ∠' at F, their Halves, are equal. *Ax.* 8. |
| Also | 11 | All the ∠. at A, B, C, D, E, are equal. *Const.* |
| And | 12 | FA = FB = FC = FD = FE. *Def.* 16. 1. |
| ∴ e. 12. 11. 10. | 13 | FA, FB, FC, FD, FE, all the ∠' at A, B, C, D, E, as also all those at F, are equal. |
| ∴ e. 13. | 14 | GA, AH, HB, BI, IC, CK, KD, DL, LE, EG, are equal. 26. 1. |
| ∴ e. 14. | 15 | GH, HI, IK, KL, LG, their Doubles, are equal. *Ax.* 7. |
| Also. e. 13. | 16 | The ten ∠' at G, H, I, K, L, are equal. 26. 1. |
| ∴ e. 16. | 17 | The five ∠' G, H, I, K, L, their Doubles, are equal. *Ax.* 7. |
| ∴ e. 15. 17. | 18 | GH = HI = IK = KL = LG; ∠G = ∠H = ∠I = ∠K = ∠L. |
| ∴ e. 18. | 19 | GHIKL is a regular Pentagon. *Def.* 6. |
| Also | 20 | All its Sides touch the Periphery of the Circle ABCDE. *Cor.* 16. 3. |
| ∴ e. 20. | 21 | **The regular Pentagon GHIKL is described about the Circle ABCDE.** *Def.* 4. Q. E. F. & D. |

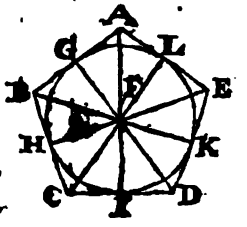
## PROP. XIII.

*In a regular Pentagon given* ABCDE, *to inſcribe a Circle* LGHIK.

### PREPARATION.

Biſect (9. 1.) any two neareſt Angles of the Pentagon A and B, with the Right-lines AF, BF, meeting in the Point F; then having from F, drawn the Perpendiculars FL, FG, FH, FI, FK, ſo that the Angles at L, G, H, I, K, may (*Def.* 10. 1.) be Right, and conſequently (*Ax.* 21.) equal, draw FC, FD, FE, and from the Center F, at the Diſtance FL, deſcribe the Circle LGHIK.

I ſay, **The Circle LGHIK is inſcribed in the regular Pentagon ABCDE.**




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | BA = BC. *Def.* 6. |
| And | 2 | BF is common; ∠FBA = ∠FBC. *Conſt.* |
| ∵ e. 1. 2. | 3 | BA = BC, BF is common, ∠FBA = ∠FBC. |
| ∵ e. 3. | 4 | ∠FAB = ∠FCB. 4. 1. |
| Again | 5 | ∠C = ∠A. *Def.* 6. |
| 5 ÷ 2 | 6 | ½∠C = ½∠A. *Ax.* 14. |
| Alſo | 7 | ∠FAB = ½∠A. *Conſt.* |
| ∵ e. 7. 6. | 8 | ∠FAB = ½∠C. *Ax.* 8. |
| ∵ e. 4. 8. | 9 | ∠FCB = ½∠C. *Ax.* 1. |
| ∵ e. 9. | 10 | ∠C is biſected. |
| Alſo | 11 | After the ſame manner is ∠D biſected. |
| ∵ | 12 | The four whole ∠' at A, B, C, D, are biſected. |
| Now | 13 | The four whole ∠' at A, B, C, D, are equal. *Def.* 6. |
| ∵ e. 13. | 14 | The eight ∠' at A, B, C, D, their Halves, are equal. *Ax.* 8. |
| Alſo | 15 | All the ∠' at L, G, H, I, K, are equal. *Conſt.* |
| And | 16 | FA, FG, FB, FH, FC, FI, FD, are common. |
| ∵ e. 16. 15. 14. | 17 | FA, FG, FB, FH, FC, FI, FD, are common, all the ∠' at L, G, H, I, K, as alſo all at A, B, C, D, are equal. |

The

∴ e. 17. 18 The eight △' from L to K, are equal. 26. 1.
∴ e. 18. 19 FL = FG = FH = FI = FK. 26. 1.
∴ e. 19. 20 The Circle LGHIK passes thro' the Points L, G, H, I, K. *Def.* 16. 1.
Also. 21 The Periphery of the Circle LGHIK, touches all the Sides of the regular Pentagon. *Cor.* 16. 3.
∴ e. 21. 22 **The Circle LGHIK is inscribed in the regular Pentagon ABCDE.** *Def.* 5. Q. E. F. & D.

## COROLLARY.

If any two nearest Angles of an Equilateral and Equiangular Figure be bisected, and from that Point in which the Lines meet that bisect the Angles, be drawn Right-lines to the remaining Angles of the Figure, all the Angles of the Figure shall be bisected.

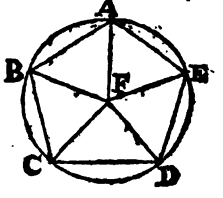
## PROP. XIV.

*About a regular Pentagon given* ABCDE, *to describe a Circle* ABCDE.

## PREPARATION.

Bisect (9. 1.) any two nearest Angles of the Pentagon, with the Right-lines AF, BF, meeting in the Point F; then from the Point F, draw the Right lines FC, FD, FE, so (*Cor.* 13.) will the Angles C, D, E, be also bisected.

I say, **The Circle ABCDE, is described about the regular Pentagon ABCDE.**




## DEMONSTRATION.

1 ∠A = ∠B, ∠B = ∠C. *Def.* 6.
∴ e. 1. 2 ∠FAB = ∠FBA, ∠FBC = ∠FCB. *Ax.* 8.
∴ e. 2. 3 FA = FB, FB = FC. 6. 1.
Again 4 ∠C = ∠D, ∠D = ∠E. *Def.* 6.
∴ e. 4. 5 ∠FCD = ∠FDC, ∠FDE = ∠FED. *Ax.* 8.
∴ e. 5. 6 FC = FD, FD = FE. 6. 1.
∴ e. 3. 6. 7 FA = FB = FC = FD = FE. *Ax.* 1.
∴ e. 7. 8 The Circumference of the Circle ABCDE passes thro' the Vertex of each of the ∠' A, B, C, D, E. *Def.* 16. 1.
∴ e. 8. 9 **The Circle ABCDE, is described about the regular Pentagon ABCDE.** *Def.* 3. Q. E. F. & D.

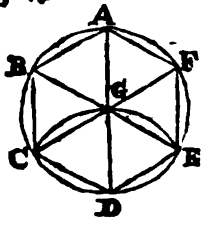
## PROP. XV.

*In a Circle given* ABCDEF, *to inſcribe a regular Hexagon* ABCDEF.

### PREPARATION.

Draw the Diameter AD, and from its extremity D, thro' the Center G, deſcribe the Arch CGE; then from the Points C, E, draw the Diameters CF, EB, and join CD, DE, EF, FA, AB, BC.

I ſay, **The regular Hexagon ABCDEF, is inſcribed in the Circle ABCDEF.**




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | DC=DG=GC; DE=DG=GE. *Def.* 16. 1. |
| ∴ e. 1. | 2 | ∠CGD=60°, ∠DGE=60°. *Cor.* 4. 32. 1. |
| Alſo | 3 | ∠CGD=∠FGA; ∠DGE=∠AGB. 15. 1. |
| ∴ e. 3. 2. | 4 | ∠FGA=60°, ∠AGB=60°. *Ax.* 1. |
| ∴ e. 4. | 5 | ∠FGA+∠AGB=120°. |
| Alſo | 6 | ∠FGA+∠AGB+∠BGC=180°. *Cor.* 2, 13. 1. |
| 6 — 5 | 7 | ∠BGC=60°. *Ax.* 12. |
| Alſo | 8 | ∠BGC=∠EGF. 15. 1. |
| ∴ e. 8. 7. | 9 | ∠EGF=60°. *Ax.* 1. |
| ∴ e. 2. 9. 4. 7. | 10 | ∠CGD=∠DGE=∠EGF=∠FGA=∠AGB=∠BGC. *Ax.* 1. |
| ∴ e. 10. | 11 | Arch CD= Arch DE= Arch EF= Arch FA= Arch AB= Arch BC. 26. 3. |
| ∴ e. 11. | 12 | CD=DE=EF=FA=AB=BC. 29. 3. |
| Again | 13 | Each of the ∠'CDE, DEF, EFA, FAB, ABC, BCD, ſtand on four of the equal Arches. |
| ∴ e. 13. | 14 | ∠CDE=∠DEF=∠EFA=∠FAB=∠ABC=∠BCD. 27. 3. |
| ∴ e. 12. 14 | 15 | CD=DE=EF=FA=AB=BC; ∠CDE=∠DEF=∠EFA=∠FAB=∠ABC=∠BCD. |
| ∴ e. 15. | 16 | ABCDE is a regular Hexagon. *Def.* 6. |
| Alſo | 17 | The Vertex of each of its Angles touches the Circumference of the Circle ABCDEF. |
| ∴ e. 17. | 18 | **The regular Hexagon ABCDEF, is inſcribed in the Circle ABCDEF.** *Def.* 2. Q. E. F. & D. |

COROL-

## COROLLARY.

The Side of a regular Hexagon inſcribed in a Circle, is equal the Semidiameter.

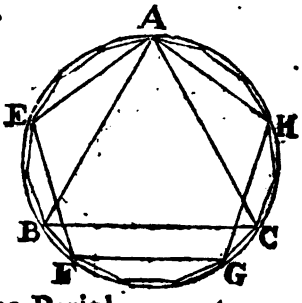
## PROP. XVI.

*In a Circle given* AEBC, *to inſcribe a regular Pentedecagon.*

## PREPARATION.

Inſcribe (2.) in the Circle AEBC, an Equilateral Triangle ABC, as alſo (11.) a regular Pentagon AEFGH, ſo that the Angles may meet at the Point A, and join BF.

I ſay, **The Pentedecagon whoſe Side is BF, is regular and inſcribed in the Circle AEBC.**




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | Arch AF $= \frac{1}{5}$ of the Periphery. *Conſt.* |
| i. e. | 2 | Arch AF $= \frac{6}{15}$ of the Periphery. |
| Alſo | 3 | Arch AB $= \frac{1}{3}$ of the Periphery. *Conſt.* |
| i. e. | 4 | Arch AB $= \frac{5}{15}$ of the Periphery. |
| 2 — 4. | 5 | Arch AF — Arch AB $= \frac{6}{15} - \frac{5}{15}$ of the Periphery. *Ax.* 12. |
| i. e. | 6 | Arch BF $= \frac{1}{15}$ of the Periphery. |
| ∴ e. 6. | 7 | The Pentedecagon whoſe Side is BF, is Equilateral. |
| Again | 8 | Each of its Angles ſtand on Thirteen of the equal Arches. |
| ∴ e. 8. | 9 | The Pentedecagon whoſe Side is BF, is Equiangular. 27. 3. |
| ∴ e. 7. 9. | 10 | The Pentedecagon whoſe Side is BF, has equal Sides and equal Angles. |
| ∴ e. 10. | 11 | The Pentedecagon whoſe Side is BF, is regular. *Def.* 6. |
| Alſo | 12 | The Vertex of each of its Angles touches the Circumference of the Circle AEBC. |
| ∴ e. 12. | 13 | **The Pentedecagon whoſe Side is BF, is regular and inſcribed in the Circle AEBC.** *Def.* 2. Q. E. F. & D. |

*The End of the Fourth Book.*

# EUCLID's ELEMENTS,

Demonstrated after a New, Plain, and Easie METHOD.

## BOOK V.

### DEFINITIONS.

I

**Ratio** (or **rate**) is the mutual Habitude or Respect of two Quantities of the same Kind each to other, according to Quantity.

In every Ratio, that Quantity which is referred to another Quantity, is called the **Antecedent** of the Ratio, and that to which the other is referred, is called the **Consequent** of the Ratio; as in the Ratio of 6 to 3, 6 is the **Antecedent**, and 3 the **Consequent**.

The Quantity of any Ratio is known by dividing the Antecedent by the Consequent; as the Ratio of 6 to 3, is expressed by $\frac{6}{3} = 2$, or the Quantity of the Ratio of A to B, is $\frac{A}{B}$.

Where-

Wherefore, often for Brevities Sake, we denote the Quantities of Ratio's, thus; $\frac{A}{B}$ ⊏ or = or ⊐ $\frac{C}{D}$. that is, the Ratio of A to B is greater, equal, or leſs than the Ratio of C to D.

2 **Proportional Quantities**, whether **Continued**, or **Diſcontinued**, are ſuch as have the ſame Ratio. Thus the **Continued** Quantities ARRR, ARR, AR, A, or 16, 8 4. 2, are proportional, becauſe the Ratio of $AR^3$ to $AR^2$, or of 16 to 8, that is, $\frac{AR^3}{AR^2} = R$, or $\frac{16}{8} = 2$, is the ſame with that of AR to A, or 4 to 2, that is, with $\frac{AR}{A} = R$, or $\frac{4}{2} = 2$. Alſo the **Diſcontinued** Quantities AR, A, BR, B, or 4, 2, 6, 3, are proportional, becauſe the Ratio, of AR to A, or of 4 to 2, that is, $\frac{AR}{A} = R$, or $\frac{4}{2} = 2$, is the ſame with that of BR to B, or of 6 to 3, that is, with $\frac{BR}{B} = R$, or $\frac{6}{3} = 2$.

Whence equal Ratio's may be expreſſed as are Proportionals, and Proportionals as are equal Ratio's.

Thus $\frac{AR}{A} = \frac{BR}{B}$, therefore AR : A :: BR : B. alſo AR : A :: BR : B. therefore $\frac{AR}{A} = \frac{BR}{B}$.

3 **Inverſe Ratio**, is when the Conſequent is taken as the Antecedent, and ſo compared to the Antecedent as the Conſequent. As if A : C :: B : C. therefore by **Inverſion**, C : A :: C : B.

4 **Alternate Proportion**, is the Comparing of Antecedent to Antecedent, and Conſequent to Conſequent.

As ARRR : ARR : : AR : A. or AR : A : : BR : B. therefore **Alternately**, or by **Permutation**, $AR^3 : AR :: AR^2 : A$. or AR : BR : : A : B.

5 **Divided Ratio**, is when the Exceſs wherein the Antecedent exceeds the Conſequent, is compared to the Conſequent.

As ARRR : ARR : : AR : A. or AR : A : : BR : B. therefore by **Diviſion**, $AR^3 - AR^2 : AR^2 :: AR - A : A$. or AR — A : A : : BR — B : B.

6 **Compound Ratio**, is when the Antecedent and Conſeqent taken both as one, are compared to the Conſequent it ſelf.

As ARRR : ARR : : AR : A. or AR : A : : BR : B. therefore by **Compoſition**, $AR^3 + AR^2 : AR^2 :: AR + A : A$. or AR + A : A : : BR + B : B.

7 **Converſe Ratio**, is when the Antecedent is compared to the Exceſs wherein the Antecedent exceeds the Conſequent.

As if AB + BC : AB : : DE + EF : DE. therefore by **Converſion**, AB + BC : BC : : DE + EF : EF.

Note: *The Six firſt Propoſitions of this Book, ſerving only for the Proof of thoſe that follow, by the Method of Equimultiples, my intent being not to uſe that Method, are here omitted.*

PROP

## PROP. VII.

*Equal Quantities* A, B, *have to the ſame Quantity* C, *the ſame Proportion; and the ſame Quantity* C, *has the ſame Proportion to equal Quantities* A, B.

That is, $\mathfrak{A} : \mathfrak{C} :: \mathfrak{B} : \mathfrak{C}$. $\mathfrak{C} : \mathfrak{A} :: \mathfrak{C} : \mathfrak{B}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $A = B$. *Hyp.* |
| And | 2 | $C = C$. |
| 1 ÷ 2. | 3 | $\frac{A}{C} = \frac{B}{C}$. *Ax.* 14. |
| 2 ÷ 1. | 4 | $\frac{C}{A} = \frac{C}{B}$. *Ax.* 14. |
| ∴ c. 3. 4. | 5 | $\mathfrak{A} : \mathfrak{C} :: \mathfrak{B} : \mathfrak{C}$. $\mathfrak{C} : \mathfrak{A} :: \mathfrak{C} : \mathfrak{B}$. *Def.* 2. Q. E. D. |

### COROLLARY.

Hence Quantities directly proportional, are (*Def.* 3.) alſo ſo by Inverſion.

## PROP. VIII.

*Of unequal Quantities* A, B, *the greater* A, *has a greater Ratio to a third* C, *than the leſſer* B; *and this third* C, *has a greater Ratio to the leſſer* B, *than the greater* A.

That is, $\frac{\mathfrak{A}}{\mathfrak{C}} \sqsubset \frac{\mathfrak{B}}{\mathfrak{C}}$. $\frac{\mathfrak{C}}{\mathfrak{B}} \sqsubset \frac{\mathfrak{C}}{\mathfrak{A}}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $A \sqsubset B$. *Hyp.* |
| ∴ c. 1. | 2 | $A \div C \sqsubset B \div C$. |
| Again | 3 | $B \sqsupset A$. *Hyp.* |
| ∴ c. 3. | 4 | $C \div B \sqsubset C \div A$. |
| ∴ c. 2. 4. | 5 | $\frac{\mathfrak{A}}{\mathfrak{C}} \sqsubset \frac{\mathfrak{B}}{\mathfrak{C}}$. $\frac{\mathfrak{C}}{\mathfrak{B}} \sqsubset \frac{\mathfrak{C}}{\mathfrak{A}}$. Q. E. D. |

## PROP. IX.

*Quantities* A, B, *which to one and the ſame Quantity* C, *have the ſame Proportion, are equal the one to the other; and if a Quantity* C, *have the ſame Proportion to other Quantities* A, B, *thoſe Quantities are equal.*

| | | |
|---|---|---|
| *Hyp.* | 1 | Suppoſe A : C :: B : C. |
| | | I ſay, $\mathfrak{A} = \mathfrak{B}$. |

### DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If A ſhould be either ⊏ or ⊐ B. |
| Then | 2 | $\frac{A}{C}$ would be either ⊏ or ⊐ $\frac{B}{C}$. 8. |
| But | 3 | $\frac{A}{C} = \frac{B}{C}$. *Hyp.* |
| ∴ | 4 | A is neither ⊏ nor ⊐ B. |
| ∴ e. 4. | 5 | $\mathfrak{A} = \mathfrak{B}$. *Ax.* 4. Q. E. D. |

| | | |
|---|---|---|
| *Hyp.* | 2 | Suppoſe C : A :: C : B. |
| | | I ſay, $\mathfrak{A} = \mathfrak{B}$. |

### DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If A ſhould be either ⊏ or ⊐ B. |
| Then | 2 | $\frac{C}{A}$ would be either ⊐ or ⊏ $\frac{C}{B}$. 8. |
| But | 3 | $\frac{C}{A} = \frac{C}{B}$. *Hyp.* |
| ∴ | 4 | A is neither ⊏ nor ⊐ B. |
| ∴ e. 4. | 5 | $\mathfrak{A} = \mathfrak{B}$. *Ax.* 4. Q. E. D. |

PROP.

## PROP. X.

*Of Quantities* A, B, *having Ratio to the ſame Quantity* C, *that Quantity* A, *which has the greater Ratio, is the greater Quantity; and that Quantity* B, *to which the ſame carries a greater Ratio, is the leſſer Quantity.*

| | | |
|---|---|---|
| *Hyp.* | 1 | Suppoſe $\frac{A}{C} \sqcap \frac{B}{C}$. |

I ſay, $\mathfrak{A} \sqsubset \mathfrak{B}$.

## DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If A ſhould be = B. |
| Then | 2 | $\frac{A}{C}$ would be $= \frac{B}{C}$. 7. |
| But | 3 | $\frac{A}{C} \sqsubset \frac{B}{C}$. *Hyp.* |
| ∴ | 4 | A is not = B. |
| Again | 5 | If A ſhould be $\sqcap$ B. |
| Then | 6 | $\frac{A}{C}$ would be $\sqcap \frac{B}{C}$. 8. |
| But | 7 | $\frac{A}{C} \sqsubset \frac{B}{C}$. *Hyp.* |
| ∴ | 8 | A is not $\sqcap$ B. |
| ∴ e. 4. 8. | 9 | A is neither = nor $\sqcap$ B. |
| ∴ e. 9. | 10 | $\mathfrak{A} \sqsubset \mathfrak{B}$. *Ax.* 5. Q. E. D. |

| | | |
|---|---|---|
| *Hyp.* | 2 | Suppoſe $\frac{C}{B} \sqsubset \frac{C}{A}$. |
| | | I ſay, $\mathfrak{B} \urcorner \mathfrak{A}$. |

DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If B ſhould be = A. |
| Then | 2 | $\frac{C}{B}$ would be $= \frac{C}{A}$. 7. |
| But | 3 | $\frac{C}{B} \sqsubset \frac{C}{A}$. *Hyp.* |
| ∴ | 4 | B is not = A. |
| Again | 5 | If B ſhould be $\sqsubset$ A. |
| Then | 6 | $\frac{C}{B}$ would be $\urcorner \frac{C}{A}$. 8. |
| But | 7 | $\frac{C}{B} \sqsubset \frac{C}{A}$. *Hyp.* |
| ∴ | 8 | B is not $\sqsubset$ A. |
| ∴ e. 4. 8. | 9 | B is neither = nor $\sqsubset$ A. |
| ∴ e. 9. | 10 | $\mathfrak{B} \urcorner \mathfrak{A}$. *Ax.* 6. Q. E. D. |

## PROP. XI.

*Proportions which are one and the ſame to any Third, are alſo the ſame to one another.*

Suppoſe A : B :: E : F. and C : D :: E : F.

I ſay, $\mathfrak{A} : \mathfrak{B} :: \mathfrak{C} : \mathfrak{D}$.

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\frac{A}{B} = \frac{E}{F}$. *Hyp.* |
| And | 2 | $\frac{C}{D} = \frac{E}{F}$. *Hyp.* |
| ∴ e. 1. 2. | 3 | $\frac{A}{B} = \frac{C}{D}$. *Ax.* 1. |
| ∴ e 3. | 4 | $\mathfrak{A} : \mathfrak{B} :: \mathfrak{C} : \mathfrak{D}$. *Def.* 2. Q. E. D. |

LEMMA.

## LEMMA I.

*If four Quantities* ARRR, ARR, AR, A, *or* AR, A, BR, B, *either continued or diſcontinued, are Proportional, that is, if* $AR^3 : AR^2 :: AR : A$. *and* $AR : A :: BR : B$. *the Product of the Extreams is equal to that of the Means.*

That is, $\mathfrak{AR}^3 \times \mathfrak{A} = \mathfrak{AR}^2 \times \mathfrak{AR}$. and $\mathfrak{AR} \times \mathfrak{B} = \mathfrak{BR} \times \mathfrak{A}$.

### DEMONSTRATION.

1 $AR^3 : AR^2 :: AR : A$. and $AR : A :: BR : B$. *Def.* 2.

Alſo 2 $\mathfrak{AR}^3 \times \mathfrak{A} = \mathfrak{AR}^2 \times \mathfrak{AR}$. and $\mathfrak{AR} \times \mathfrak{B} = \mathfrak{BR} \times \mathfrak{A}$. Q. E. D.

## LEMMA II.

*Thoſe four Quantities* A, B, C, D, *are Proportional, the Product of whoſe Extreams is equal to that of the Means.*

### PREPARATION.

Suppoſe $AD = BC$, and the Ratio of A to $B = X$, that is, $\frac{A}{B} = X$, then is $A = BX$; alſo ſuppoſe the Ratio of C to $D = Z$, that is, $\frac{C}{D} = Z$, then is $C = DZ$.

I ſay, $\mathfrak{A} : \mathfrak{B} :: \mathfrak{C} : \mathfrak{D}$.

### DEMONSTRATION.

1 $AD = BC$. *Hyp.*

And 2 $A = BX$.

∵ e. 1. 2. 3 $DBX = BC$. *Sub.*

Again 4 $C = DZ$.

∵ e. 3. 4. 5 $DBX = DBZ$. *Sub.*

5 ÷ DB. 6 $X = Z$. *Ax.* 14.

Laſtly 7 $\frac{A}{B} = X$, $\frac{C}{D} = Z$. *Hyp.*

∵ e. 6. 7. 8 $\frac{A}{B} = \frac{C}{D}$. *Sub.*

∵ e. 8. 9 $\mathfrak{A} : \mathfrak{B} :: \mathfrak{C} : \mathfrak{D}$. *Def.* 2. Q. E. D.

## PROP. XII.

*If ſeveral Quantities* A, AR, ARR, ARRR, ARRRR, *&c. or* A, AR, B, BR, C, CR, *&c. either continued or diſcontinued, be Proportional, that is, if* $A : AR :: AR^2 : AR^3 :: AR^4 : AR^5$. and $A : AR :: B : BR :: C : CR$, *as any one Antecedent is to its Conſequent, ſo is the Sum of all the Antecedents, to the Sum of all the Conſequents.*

That is, $\mathfrak{A} : \mathfrak{AR} :: \mathfrak{A} + \mathfrak{AR}^2 + \mathfrak{AR}^4 : \mathfrak{AR} + \mathfrak{AR}^3 + \mathfrak{AR}^5$. and $\mathfrak{A} : \mathfrak{AR} :: \mathfrak{A} + \mathfrak{B} + \mathfrak{C} : \mathfrak{AR} + \mathfrak{BR} + \mathfrak{CR}$.

### DEMONSTRATION.

1 $A : AR :: AR^2 : AR^3 :: AR^4 : AR^5$. and $A : AR :: B : BR :: C : CR$. *Def.* 2.

And 2 $\overline{A + AR^2 + AR^4} \times AR = \overline{AR + AR^3 + AR^5} \times A$. and $\overline{A + B + C} \times AR = \overline{AR + BR + CR} \times A$.

Cor. 2. 3 $\mathfrak{A} : \mathfrak{AR} :: \mathfrak{A} + \mathfrak{AR}^2 + \mathfrak{AR}^4 : \mathfrak{AR} + \mathfrak{AR}^3 + \mathfrak{AR}^5$. and $\mathfrak{A} : \mathfrak{AR} :: \mathfrak{A} + \mathfrak{B} + \mathfrak{C} : \mathfrak{AR} + \mathfrak{BR} + \mathfrak{CR}$. *Lem.* 2. Q. E. D.

## PROP. XIII.

*If two Ratio's be equal, and the one greater than a third Ratio, the other alſo will be greater than the ſame third Ratio.*

Suppoſe $\frac{A}{B} = \frac{C}{D}$, alſo $\frac{C}{D} \sqsubset \frac{E}{F}$.

I ſay, $\frac{\mathfrak{A}}{\mathfrak{B}} \sqsubset \frac{\mathfrak{E}}{\mathfrak{F}}$.

### DEMONSTRATION.

1 $\frac{A}{B} = \frac{C}{D}$. *Hyp.*

And 2 $\frac{C}{D} \sqsubset \frac{E}{F}$. *Hyp.*

Cor. 1. 2. 3 $\frac{\mathfrak{A}}{\mathfrak{B}} \sqsubset \frac{\mathfrak{E}}{\mathfrak{F}}$; *Ax.* 3. Q. E. D.

PROP.

## PROP. XIV.

*In four proportional Quantities* A, B, C, D, *if the first* A, *be greater, equal, or leſs than the third* C, *then will the ſecond* B, *be greater, equal, or leſs than the fourth* D.

*Hyp.* 1 Suppoſe $A \sqsubset C$.

I ſay, $\mathfrak{B} \sqsubset \mathfrak{D}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $A \sqsubset C$. *Hyp.* |
| ∵ c. 1. | 2 | $\frac{A}{B} \sqsubset \frac{C}{B}$. 8. |
| Alſo | 3 | $\frac{A}{B} = \frac{C}{D}$. *Hyp.* |
| ∵ c. 2. 3. | 4 | $\frac{C}{D} \sqsubset \frac{C}{B}$. *Ax.* 3. |
| ∵ c. 4. | 5 | $\mathfrak{B} \sqsubset \mathfrak{D}$. 10. Q. E. D. |

*Hyp.* 2 Suppoſe $A = C$.

I ſay, $\mathfrak{B} = \mathfrak{D}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $A = C$. *Hyp.* |
| ∵ c. 1. | 2 | $\frac{A}{B} = \frac{C}{B}$. 7. |
| Alſo | 3 | $\frac{A}{B} = \frac{C}{D}$. *Hyp.* |
| ∵ c. 2. 3. | 4 | $\frac{C}{B} = \frac{C}{D}$. *Ax.* 1 |
| ∵ c. 4. | 5 | C : B :: C : D. *Def.* 2. |
| ∵ c. 5. | 6 | $\mathfrak{B} = \mathfrak{D}$. 9. Q. E. D. |

| | | |
|---|---|---|
| *Hyp.* | 3 | Suppoſe $A \sqsupset C$. |
| | | I ſay, $\mathfrak{B} \sqsupset \mathfrak{D}$. |

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $A \sqsupset C$. *Hyp.* |
| ∵ c. 1. | 2 | $\frac{A}{B} \sqsupset \frac{C}{B}$. 8. |
| Alſo | 3 | $\frac{A}{B} = \frac{C}{D}$. *Hyp.* |
| ∵ c. 2. 3. | 4 | $\frac{C}{D} \sqsupset \frac{C}{B}$. *Ax.* 3. |
| ∵ c. 4. | 5 | $\mathfrak{B} \sqsupset \mathfrak{D}$. 10. Q. E. D. |

## PROP. XV.

*If of four Quantities* AR, A, BR, B, *the ſecond* A, *be as often contained in the firſt* AR, *as is the fourth* B, *in the third* BR, *that is, if* $\frac{AR}{A}$ *be* $= \frac{BR}{B}$, *thoſe four Quantities are Proportional.*

That is, $\mathfrak{AR} : \mathfrak{A} :: \mathfrak{BR} : \mathfrak{B}$.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\frac{AR}{A} = \frac{BR}{B}$. *Def.* 2. |
| ∵ c. 1. | 2 | $\mathfrak{AR} : \mathfrak{A} :: \mathfrak{BR} : \mathfrak{B}$. *Def.* 2. Q. E. D. |

PROP.

## PROP. XVI.

*If four Quantities* ARRR, ARR, AR, A, *or* AR, A, BR, B, *either continued or discontinued, be Proportional, that is, if* $AR^3 : AR^2 :: AR : A$. *and* $AR : A :: BR : B$, *they are also so when Altern'd.*

That is, $\mathfrak{AR}^3 : \mathfrak{AR} :: \mathfrak{AR}^2 : \mathfrak{A}$. and $\mathfrak{AR} : \mathfrak{BR} :: \mathfrak{A} : \mathfrak{B}$.

## DEMONSTRATION.

1 $AR^3 : AR^2 :: AR : A$. and $AR : A :: BR : B$. *Def.* 2.
c. 1. 2 $AR^3 \times A = AR \times AR^2$. and $AR \times B = BR \times A$. *Lem.* 1.
c. 2. 3 $\mathfrak{AR}^3 : \mathfrak{AR} :: \mathfrak{AR}^2 : \mathfrak{A}$. and $\mathfrak{AR} : \mathfrak{BR} :: \mathfrak{A} : \mathfrak{B}$. *Lem.* 2.
Q. E. D.

## PROP. XVII.

*If four Quantities* ARRR, ARR, AR, A, or AR, A, BR, B, *either continued or discontinued, be Proportional, that is, if* $AR^3 : AR^2 :: AR : A$. *and* $AR : A :: BR : B$, *they are also so when Divided.*

That is, $\mathfrak{AR}^3 - \mathfrak{AR}^2 : \mathfrak{AR}^2 :: \mathfrak{AR} - \mathfrak{A} : \mathfrak{A}$. and $\mathfrak{AR} - \mathfrak{A} : \mathfrak{A} :: \mathfrak{BR} - \mathfrak{B} : \mathfrak{B}$.

## DEMONSTRATION.

1 $AR^3 : AR^2 :: AR : A$. and $AR : A :: BR : B$. *Def.* 2.
And 2 $\overline{AR^3 - AR^2} \times A = \overline{AR - A} \times AR^2$. and $\overline{AR - A} \times B = \overline{BR - B} \times A$.
c. 2. 3 $\mathfrak{AR}^3 - \mathfrak{AR}^2 : \mathfrak{AR}^2 :: \mathfrak{AR} - \mathfrak{A} : \mathfrak{A}$. and $\mathfrak{AR} - \mathfrak{A} : \mathfrak{A} :: \mathfrak{BR} - \mathfrak{B} : \mathfrak{B}$. *Lem.* 2. Q. E. D.

## PROP. XVIII.

*If four Quantities* ARRR, ARR, AR, A, *or* AR, A, BR, B, *continued or diſcontinued, are Proportional, that is, if* $AR^3 : AR^2 :: AR : A$. *and* $AR : A :: BR : B$. *they are alſo ſo when Compounded.*

That is, $\mathfrak{AR}^3 + \mathfrak{AR}^2 : \mathfrak{AR}^2 :: \mathfrak{AR} + \mathfrak{A} : \mathfrak{A}$, and $\mathfrak{AR} + \mathfrak{A} : \mathfrak{A} :: \mathfrak{BR} + \mathfrak{B} : \mathfrak{B}$.

### DEMONSTRATION.

1 $AR^3 : AR^2 :: AR : A$. and $AR : A :: BR : B$. *Def.* 2.

And 2 $\overline{AR^3 + AR^2} \times A = \overline{AR + A} \times AR^2$. and $\overline{AR + A} \times B = \overline{BR + B} \times A$.

∴ e. 2. 3 $\mathfrak{AR}^3 + \mathfrak{AR}^2 : \mathfrak{AR}^2 :: \mathfrak{AR} + \mathfrak{A} : \mathfrak{A}$. and $\mathfrak{AR} + \mathfrak{A} : \mathfrak{A} :: \mathfrak{BR} + \mathfrak{B} : \mathfrak{B}$. *Lem.* 2. Q. E. D.

## PROP. XIX.

*If the whole* AB + BC, *be to the whole* DE + EF, *as the Part taken away* AB, *is to the Part taken away* DE, *then ſhall the Reſidue* BC, *be to the Reſidue* EF, *as the whole* AB + BC, *is to the whole* DE + EF.

That is, $\mathfrak{BC} : \mathfrak{EF} :: \mathfrak{AB} + \mathfrak{BC} : \mathfrak{DE} + \mathfrak{EF}$.

### DEMONSTRATION.

1 AB + BC : DE + EF :: AB : DE. *Hyp.*

∴ e. 1. 2 AB + BC : AB :: DE + EF : DE. 16.

∴ e. 2. 3 AB + BC — AB : AB :: DE + EF — DE : DE. 17.

i. e. 4 BC : AB :: EF : DE.

∴ e. 4. 5 BC : EF :: AB : DE. 16.

∴ e. 5. 1. 6 $\mathfrak{BC} : \mathfrak{EF} :: \mathfrak{AB} + \mathfrak{BC} : \mathfrak{DE} + \mathfrak{EF}$. 11. Q. E. D.

COROL-

## COROLLARY.

Becauſe (*Step.* 2.) 'tis AB+BC:AB::DE+EF:DE. and by inverting the Sixth Step, 'tis AB+BC:BC::DE+EF:EF. therefore Quantities directly Proportional, are (*Def.* 7.) alſo ſo by Converſion.

## PROP. XX.

*If there be three Quantities* A, B, C, *and others* D, E, F, *equal to them in Number, which being taken two and two are in the ſame Ratio, and their Proportion Ordinate, that is, if* A:B::D:E. *and* B:C::E:F. *and if the firſt* A *be greater, equal, or leſs than the third* C, *then ſhall the fourth* D *be greater, equal, or leſs than the ſixth* F.

*Hyp.* 1 Suppoſe A ⊏ C.

I ſay, D ⊏ F.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | B:C::E:F. *Hyp.* |
| ∴ e. 1. | 2 | C:B::F:E. *Cor.* 7. |
| ∴ e. 2. | 3 | $\frac{C}{B} = \frac{F}{E}$. *Def.* 2. |
| Again | 4 | A ⊏ C. *Hyp.* |
| ∴ e. 4. | 5 | $\frac{A}{B} \sqsubset \frac{C}{B}$. 8. |
| ∴ e. 3. 5. | 6 | $\frac{A}{B} \sqsubset \frac{F}{E}$. *Ax.* 2. |
| Alſo | 7 | $\frac{A}{B} = \frac{D}{E}$. *Hyp.* |
| ∴ e. 6. 7. | 8 | $\frac{D}{E} \sqsubset \frac{F}{E}$. *Ax.* 3. |
| ∴ e. 8. | 9 | D ⊏ F, 10. Q. E. D. |

| | | |
|---|---|---|
| *Hyp.* | 2 | Suppose A = C. |
| | | I say, ℬ = ℱ. |

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | B : C :: E : F. *Hyp.* |
| And | 2 | A = C. *Hyp.* |
| ∵ e. 1. 2. | 3 | B : A :: E : F. *Sub.* |
| ∵ e. 3. | 4 | A : B :: F : E. *Cor.* 7. |
| Alſo | 5 | A : B :: D : E. *Hyp.* |
| ∵ e. 3. 4. | 6 | D : E :: F : E 11. |
| ∵ e. 6. | 7 | ℬ = ℱ. 9. Q. E. D. |

| | | |
|---|---|---|
| *Hyp.* | 3 | Suppoſe A ⌝ C. |
| | | I ſay, ℬ ⌝ ℱ. |

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | B : C :: E : F. *Hyp.* |
| ∵ e. 1. | 2 | C : B :: F : E. *Cor.* 7. |
| ∵ e. 2. | 3 | $\frac{C}{B} = \frac{F}{E}$. *Def.* 2. |
| | | A ⌝ C. *Hyp.* |
| Again | 4 | $\frac{A}{B} \urcorner \frac{C}{B}$. 8. |
| ∵ e. 4. | 5 | $\frac{A}{B} \urcorner \frac{F}{E}$. *Ax.* 2. |
| ∵ e. 3. 5. | 6 | $\frac{A}{B} = \frac{D}{E}$. *Hyp.* |
| Alſo | 7 | $\frac{D}{E} \urcorner \frac{F}{E}$. *Ax.* 3. |
| ∵ e. 6. 7. | 8 | ℬ ⌝ ℱ. 10. Q. E. D. |
| ∵ e. 8. | 9 | |

PROP.

## PROP. XXI.

*If there be three Quantities* A, B, C, *and others* D, E, F, *equal to them in Number, which taken two and two are in the ſame Ratio, and their Proportion Inordinate; that is, if* A : B :: E : F. *and* B : C :: D : E. *and if the firſt* A *be greater, equal, or leſs than the third* C, *then is the fourth* D *greater, equal, or leſs than the ſixth* F.

*Hyp.* 1 Suppoſe A ⊏ C.

I ſay, ✠ ⊏ ✠.

### DEMONSTRATION.

1 B : C :: D : E. *Hyp.*

∵ c. 1. 2 C : B :: E : D. *Cor.* 7.

∵ c. 2. 3 $\frac{C}{B} = \frac{E}{D}$. *Def.* 20.

Again 4 A ⊏ C. *Hyp.*

∵ c. 4. 5 $\frac{A}{B} \sqsubset \frac{C}{B}$. 8.

∵ c. 3. 5. 6 $\frac{A}{B} \sqsubset \frac{E}{D}$. *Ax.* 2.

Alſo 7 $\frac{A}{B} = \frac{E}{F}$. *Hyp.*

∵ c. 8. 7. 8 $\frac{E}{F} \sqsubset \frac{E}{D}$. *Ax.* 3.

∵ c. 8. 9 ✠ ⊏ ✠. 10. Q. E. D.

*Hyp.* 2 Suppoſe A = C.

I ſay, ✠ = ✠.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | B : C :: D : E. *Hyp.* |
| ∵ c. 1. | 2 | C : B :: E : D. *Cor.* 7. |
| Again | 3 | A = C. *Hyp.* |
| ∵ c. 2. 3. | 4 | A : B :: E : D. *Sub.* |
| Alſo | 5 | A : B :: E : F. *Hyp.* |
| ∵ c. 4. 5. | 6 | E : D :: E : F. 11. |
| ∵ c. 6. | 7 | $\frac{E}{D} = \frac{E}{F}$. *Def.* 2. |
| ∵ c. 7. | 8 | D = F. 9. Q. E. D. |
| *Hyp.* | 3 | Suppoſe A ⌝ C. |
| | | I ſay, D ⌝ F. |

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | B : C :: D : E. *Hyp.* |
| ∵ c. 1. | 2 | C : B :: E : D. *Cor.* 7. |
| ∵ c. 2. | 3 | $\frac{C}{B} = \frac{E}{D}$. *Def.* 2. |
| Again | 4 | A ⌝ C. *Hyp.* |
| ∵ c. 4. | 5 | $\frac{A}{B}$ ⌝ $\frac{C}{B}$. 8. |
| ∵ c. 3. 5. | 6 | $\frac{A}{B}$ ⌝ $\frac{E}{D}$. *Ax.* 2. |
| Alſo | 7 | $\frac{A}{B} = \frac{E}{F}$. *Hyp.* |
| ∵ c. 6. 7. | 8 | $\frac{E}{F}$ ⌝ $\frac{E}{D}$. *Ax.* 3. |
| ∵ c. 8. | 9 | D ⌝ F. 10. Q. E. D. |

PROP.

## PROP. XXII.

*If there be a certain Number of Quantities* A, B, C, *in one Rank, and others* D, E, F, *equal to them in Number in another, which taken two and two are in the same Ratio, and their Proportion Ordinate, that is, if* A : B :: D : E. *and* B : C :: E : F. *the Ratio of the two Extreams of the one Rank, shall be equal to the Ratio of the two Extreams of the other.*

That is, A : C :: D : F.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | B : C :: E : F. *Hyp.* |
| ∴ c. 1. | 2 | B : E :: C : F. 16. |
| Also | 3 | A : B :: D : E. *Hyp.* |
| ∴ c. 3. | 4 | A : D :: B : E. 16. |
| ∴ c. 4. 2. | 5 | A : D :: C : F. 11. |
| ∴ c. 5. | 6 | A : C :: D : F. 16. Q. E. D. |

## PROP. XXIII.

*If there be a certain Number of Quantities* A, B, C, *in one Rank, and others* D, E, F, *equal to them in Number in another, which taken two and two are in the same Ratio, and their Proportion Inordinate; that is, if* A : B :: E : F. *and* B : C :: D : E. *the Ratio of the two Extreams of the one Rank, is equal to the Ratio of the two Extreams of the other.*

That is, A : C :: D : F.

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $A : B :: E : F$. *Hyp.* |
| ∵ e. 1. | 2 | $AF = BE$. *Lem.* 1. |
| Alſo | 3 | $B : C :: D : E$. *Hyp.* |
| ∵ e. 3. | 4 | $CD = BE$. *Lem.* 1. |
| ∵ e. 2. 4. | 5 | $AF = CD$. *Ax.* 1. |
| ∵ e. 5. | 6 | $\mathfrak{A} : \mathfrak{C} :: \mathfrak{D} : \mathfrak{F}$. *Lem.* 2. Q. E. D. |

## PROP. XXIV.

*If of Six Quantities* A, B, C, D, E, F, *the firſt* A, *have the ſame Ratio to the ſecond* B, *which the third* C, *has to the fourth* D, *that is, if* $\frac{A}{B}$ *be* $= \frac{C}{D}$; *and the fifth* E, *have the ſame Ratio to the ſecond* B, *which the ſixth* F, *has to the fourth* D, *that is, if* $\frac{E}{B}$ *be* $= \frac{F}{D}$; *then ſhall the firſt compounded with the fifth, have the ſame Ratio to the ſecond, which the third compounded with the ſixth has to the fourth.*

That is, $\mathfrak{A} + \mathfrak{E} : \mathfrak{B} :: \mathfrak{C} + \mathfrak{F} : \mathfrak{D}$.

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\frac{A}{B} = \frac{C}{D}$. *Hyp.* |
| And | 2 | $\frac{E}{B} = \frac{F}{D}$. *Hyp.* |
| 1 + 2. | 3 | $\frac{A+E}{B} = \frac{C+F}{D}$. *Ax.* 11. |
| ∵ e. 3. | 4 | $\mathfrak{A} + \mathfrak{E} : \mathfrak{B} :: \mathfrak{C} + \mathfrak{F} : \mathfrak{D}$. *Def.* 2. Q. E. D. |

PROP.

## PROP. XXV.

*If four Quantities* A, B, C, D, *are proportional, that is, if* A : B :: C : D, *the Sum* A + D, *of the greateſt* A, *and the leaſt* D, *ſhall be greater than the Sum* B + C, *of the other two.*

That is, 𝔄 + 𝔇 ⊏ 𝔅 + ℭ.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | A : B :: C : D. *Hyp.* |
| ∵ e. 1. | 2 | A — B : B :: C — D : D. 17. |
| ∵ e. 2. | 3 | B : A — B :: D : C — D. *Cor.* 7. |
| Now | 4 | B ⊏ D. *Hyp.* |
| ∵ e. 3. 4. | 5 | A — B ⊏ C — D. 14. |
| 5 + B + D | 6 | A — B + B + D ⊏ C — D + B + D. *Ax.* 15. |
| i. e. | 7 | 𝔄 + 𝔇 ⊏ 𝔅 + ℭ. Q. E. D. |

*The End of the Fifth Book.*

# EUCLID's ELEMENTS,

Demonſtrated after a New, Plain, and Eaſie METHOD.

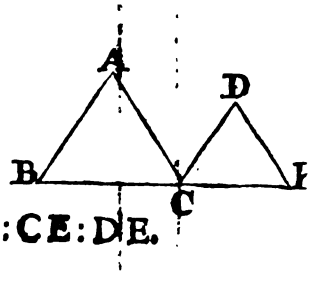
## BOOK VI.

### DEFINITIONS.

I

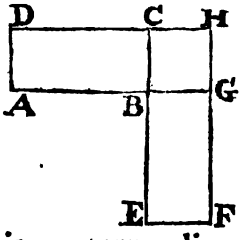
**Similar Right-lined Figures**, are ſuch whoſe ſeveral Angles are equal one to the other, and which have alſo their Sides about the equal Angles, proportional.

Thus the two Right-lined Figures ABC, DCE, are **Similar**, becauſe ∠B = ∠DCE, and AB : BC :: DC : CE; the ∠A = ∠D, and AB : AC :: DC : DE; the ∠ACB = ∠E, and BC : AC :: CE : DE.



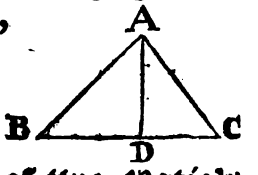

Reci-

2 **Reciprocal Figures** are such, as have Sides that may be so compared, as that the Antecedent of one Ratio, and the Consequent of the other, is to be found in the same Figure.




Thus the two Figures BD, BF, are **Reciprocal**, because as the Side AB, is to the Side BG, so is the Side EB, to the Side BC.

3 The **Altitude** of any Figure ABC, is a perpendicular Line AD, drawn from the Top A, to the Base BC.



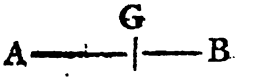

4 **A Ratio is said to be compounded of two Ratio's,** when the Quantities of the Ratio's being multiplyed the one into the other, do produce any Ratio.

Thus the Ratio of A to C, that is $\frac{A}{C}$ is compounded of the Ratio's of A to B, and B to C, that is, of $\frac{A}{B}$ and $\frac{B}{C}$.

Because $\frac{A}{B} \times \frac{B}{C} = \frac{AB}{BC} = \frac{A}{C}$.

5 In any given Quantities A, B, C, D, the Ratio of the Extreams A, D, that is $\frac{A}{D}$, is compounded of all the intermediate ones $\frac{A}{B}, \frac{B}{C}, \frac{C}{D}$. For $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{ABC}{BCD} = \frac{A}{D}$.

6 **A Right-line AB, is said to be cut according to extream and mean Ratio,** when as the whole AB, is to the greater Segment AG, so is the greater Segment AG, to the less GB.

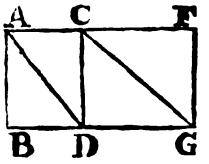


## PROPOSITION I.

*Parallelograms and Triangles, which have the ſame Height, are in Proportion one to the other, as their Baſes.*

*Caſe* 1 If the ▭ᵉ ABDC, CDGF, and △ᵉ ABD, GDC, be Right-angled.

I ſay, BD:DG::▭ABDC:▭CDGF::△ABD:△CDG.




DEMONSTRATION.

1 BD×AB=▭ABDC; DG×CD=▭CDGF. *Lem.* 1. 1.

And 2 AB=CD. 34. 1.

∵ e. 1. 2. 3 BD×AB=▭ABDC; DG×AB=▭CDGF. *Sub.*

Again 4 BD:DG::BD×AB:DG×AB. *Lem.* 2. 5.

∴ e. 4. 3. 5 BD:DG::▭ABDC:▭CDGF. *Sub.*

Laſtly 6 ▭ABDC=2△ABD; ▭CDGF=2△CDG. 41. 1.

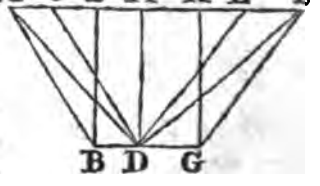
∴ e. 5. 6. 7 BD:DG::2△ABD:2△CDG. *Sub.*

∵ e. 7. 8 BD:DG::△ABD:△CDG.

∴ e. 5. 8. 9 BD:DG::▭ABDC:▭CDGF::△ABD:△CDG. 11. 5. Q. E. D.

*Caſe.* 2 If the ▭ᵉ ABDC, LDGF, and △ᵉ ABD, FDG, be Oblique-angled.

Erect Right-angled ▭ᵉ BDHI, DGKH, upon the ſame Baſes, between the ſame Parallels, or in the ſame Height with them; ſo (35. 1.) will thoſe Right-angled ▭ᵉ be equal to the Oblique-Angled ones.

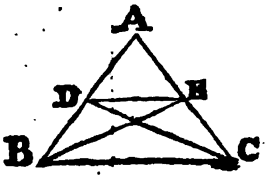



Since therefore the Right-angled ▭ᵉ BDHI, DGKH, are (*Caſe* 1.) in Proportion one to another, as their Baſes; the Oblique-angled ▭ᵉ ABDC, LDGF, their Equals; as alſo the Oblique-angled △ᵉ ABD, FDG, (which (41. 1.) are the Halves of the Right-angled ▭ᵉ BDHI, DGKH) will be in Proportion one to another, as their Baſes. Q. E. D.

PROP.

## PROP. II.

*If to one Side* BC, *of a Triangle* ABC, *be drawn a parallel Right-line* DE, *the same shall cut the Sides of the Triangle proportionally, so that* AD : DB :: AE : EC. *And if the Sides of the Triangle be cut proportionally, that is, if* AD : DB :: AE : EC. *then a Right-line* DE, *joined at the Sections* D, E, *shall be parallel to the remaining Side of the Triangle* BC.




*Hyp.* 1 Suppose DE ‖ BC.

I say, 𝔄𝔇 : 𝔇𝔅 :: 𝔄𝔈 : 𝔈𝔈.

DEMONSTRATION.

1 AD : DB :: △ADE : △BDE. 1.
And 2 △BDE = △CDE. 37. 1.
∴ e. 1. 2. 3 AD : DB :: △ADE : △CDE. *Sub.*
Also 4 AE : EC :: △ADE : △CDE. 1.
∴ e. 3. 4. 5 𝔄𝔇 : 𝔇𝔅 :: 𝔄𝔈 : 𝔈𝔈. 11. 5. Q. E. D.

*Hyp.* 2 Suppose AD : DB :: AE : EC.

I say, 𝔇𝔈 ‖ 𝔅𝔈.

DEMONSTRATION.

1 AD : DB :: AE : EC. *Hyp.*
Also 2 AD : DB :: △ADE : △BDE. 1.
∴ e. 1. 2. 3 AE : EC :: △ADE : △BDE. 11. 5.
Also 4 AE : EC :: △ADE : △CDE. 1.
∴ e. 3. 4. 5 △ADE : △BDE :: △ADE : △CDE. 11. 5.
∴ e. 5. 6 △BDE = △CDE. 9. 5.
∴ e. 6. 7 𝔇𝔈 ‖ 𝔅𝔈. 39. 1. Q. E. D.

## SCHOLIUM.

If there be drawn many Parallels to One Side of any Triangle, then all the Segments of the Sides shall be Proportional; as is easily deducible from the precedent.

## PROP. III.

*If an Angle* B A C, *of a Triangle* B A C, *be bisected, and the Right-line* A D, *that bisects the Angle, cut the Base also; then shall the Segments of the Base have the same Ratio that the other Sides of the Triangle have, so that* B D : D C :: B A : A C. *And if the Segments of the Base have the same Ratio, that the other Sides of the Triangle have, that is, if* B D : D C :: B A : A C, *then a Right-line* A D, *drawn from the Top* A, *to the Section* D, *shall bisect that Angle* B A C *of the Triangle.*

Hyp. 1 Suppose ∠ B A D = ∠ C A D.

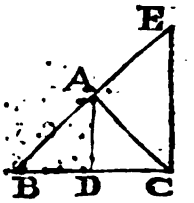
### PREPARATION.

Produce B A; and making A E = A C, join E C.

I say, B D : D C :: B A : A C.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | A E = A C. *Const.* |
| ∵ e. 1. | 2 | ∠ E = ∠ A C E. 5. 1. |
| Again | 3 | ∠ B A C = ∠ E + ∠ A C E. 32. 1. |
| ∵ e. 2. 3. | 4 | ∠ B A C = ∠ A C E + ∠ A C E. *Sub.* |
| i. e. | 5 | ∠ B A C = 2 ∠ A C E. |
| 5 ÷ 2 | 6 | ½ ∠ B A C = ∠ A C E. *Ax.* 14. |
| Also | 7 | ½ ∠ B A C = ∠ C A D. *Hyp.* |




∠ A

| | | |
|---|---|---|
| ∵ e. 6. 7. | 8 | ∠ A C E = ∠ C A D. *Ax.* 8. |
| ∵ e. 8. | 9 | A D ‖ B C. 27. 1. |
| ∵ e. 9. | 10 | B D : D C : : B A : A E. 2. |
| ∵ e. 10. 1. | 11 | B D : D C : : B A : A C. *Sub.* Q. E. D. |
| *Hyp.* | 2 | Suppose B D : D C : : B A : A C. |

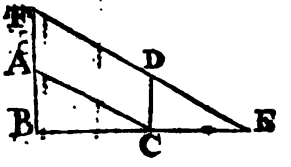
I say, ∠ B A D = ∠ C A D.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | A C = A E. *Conſt.* |
| ∵ e. 1. | 2 | ∠ A C E = ∠ E. 5. 1. |
| Again | 3 | B D : D C : : B A : A C. *Hyp.* |
| ∵ e. 1. 3. | 4 | B D : D C : : B A : A E. *Sub.* |
| ∵ e. 4. | 5 | A D ‖ E C. 2. |
| ∵ e. 5. | 6 | ∠ B A D = ∠ E. 29. 1. |
| ∵ e. 2. 6. | 7 | ∠ A C E = ∠ B A D. *Ax.* 1. |
| Alſo e. 5. | 8 | ∠ A C E = ∠ C A D. 29. 1. |
| ∵ e. 7. 8. | 9 | ∠ B A D = ∠ C A D. *Ax.* 1. Q. E. D. |

## PROP. IV.

*Of Equiangular Triangles* A B C, D C E, *the Sides which are about the equal Angles* A C B *and* D E C, B *and* D C E, B A C *and* C D E, *are proportional.*

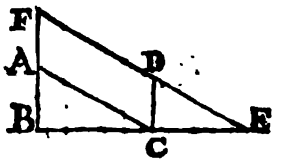



### PREPARATION.

Set the Side B C, in a direct Line to the Side C E, and produce B A and D E till they meet in F.

I say, B C : A C : : C E : D E. B C : B A : : C E : C D. B A : A C : : C D : D E.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | $\angle B = \angle DCE$, $\angle ACB = \angle DEC$. *Hyp.* |
| ∵ e. 1. | 2 | $BF \parallel CD$, $AC \parallel FE$. 28. 1. |
| ∵ e. 2. | 3 | ACDF is a ▭. *Def.* 35. 1. |
| ∵ e. 3. | 4 | $AC = FD$, $AF = CD$. 34. 1. |
| Alſo e. 2. | 5 | $BC : CE :: FD : DE$; $BC : CE :: BA : AF$. 2. |
| ∵ e. 4. 5. | 6 | $BC : CE :: AC : DE$; $BC : CE :: BA : CD$. *Sub.* |
| ∵ e. 6. | 7 | $BC : AC :: CE : DE$; $BC : BA :: CE : CD$. 16. 5. |
| Alſo e. 6. | 8 | $AC : DE :: BA : CD$. 11. 5. |
| ∵ e. 8. | 9 | $BA : AC :: CD : DE$. *Cor.* 7. 5. |
| ∵ e. 7. 9. | 10 | $\mathfrak{BC} : \mathfrak{AC} :: \mathfrak{CE} : \mathfrak{DE}$. $\mathfrak{BC} : \mathfrak{BA} :: \mathfrak{CE} : \mathfrak{CD}$. $\mathfrak{BA} : \mathfrak{AC} :: \mathfrak{CD} : \mathfrak{DE}$. Q. E. D. |

## SCHOLIA.

1 Becauſe Equiangular Triangles have alſo their Sides about the equal Angles proportional; therefore Equiangular Triangles are likewiſe *(Def.* 1.) Similar.

2 Becauſe *(Hyp.)* $\angle ACB = \angle DEC$, $\angle B$ is common, and conſequently *(Cor.* 2. 32. 1.) $\angle BAC = \angle F$; therefore if in a Triangle FBE, there be drawn AC, a parallel to one Side FE, the Triangle ABC, ſhall *(Scho.* 1.) be Similar to the whole FBE.

3 Becauſe *(Step.* 6.) $BC : CE :: AC : DE :: BA : CD$. therefore the Sides ſubtended under the equal Angles are Homologous, or of like Ratio.

PROP.

## PROP. V.

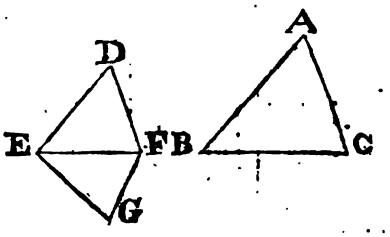
*If two Triangles* ABC, DEF, *have their Sides proportional, that is, if* AB:BC::DE:EF, *and* AC:BC::DF:EF, *and* AB:AC::DE:DF, *those Triangles are Equiangular, and have those Angles equal, under which are subtended the Homologous Sides.*

## PREPARATION.

At the Side EF, make (23. 1.) ∠GEF = ∠B, ∠GFE = ∠C, whence (*Cor.* 2. 32. 1.) ∠G = ∠A, and consequently the Triangles ABC, GEF, are Equiangular.

I say, ∠𝔄 = ∠𝔇. ∠𝔅 = ∠𝔇𝔈𝔉. ∠ℭ = ∠𝔇𝔉𝔈.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AB:BC::GE:EF. 4. |
| Also | 2 | AB:BC::DE:EF. *Hyp.* |
| ∵ e. 1. 2. | 3 | GE:EF::DE:EF. 11. 5. |
| ∵ e. 3. | 4 | GE = DE. 9. 5. |
| Again | 5 | AC:BC::GF:EF. 4. |
| Also | 6 | AC:BC::DF:EF. *Hyp.* |
| ∵ e. 5. 6. | 7 | GF:EF::DF:EF. 11. 5. |
| ∵ e. 7. | 8 | GF = DF. 9. 5. |
| Lastly | 9 | EF is common. |
| ∵ e. 4. 8. 9. | 10 | GE = DE, GF = DF, EF is common. |
| ∵ e. 10. | 11 | ∠G = ∠D, ∠GEF = ∠DEF, ∠GFE = ∠DFE. 8. 1. |
| Also | 12 | ∠G = ∠A, ∠GEF = ∠B, ∠GFE = ∠C. *Const.* |
| ∵ e. 12. 11. | 13 | ∠𝔄 = ∠𝔇. ∠𝔅 = ∠𝔇𝔈𝔉. ∠ℭ = ∠𝔇𝔉𝔈. *Ax.* 1. |

Q. E. D.

## PROP. VI.

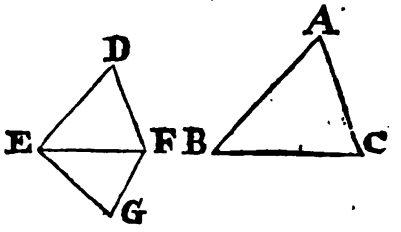
*If two Triangles* ABC, DEF, *have one Angle* B, *equal to one Angle* DEF, *and the Sides about the equal Angles* B, DEF, *proportional, that is, if* AB:BC::DE:EF; *then thoſe Triangles* ABC, DEF, *are Equiangular, and have thoſe Angles equal, under which are ſubtended the Homologous Sides.*

## PREPARATION.

At the Side EF, make (23. 1.) ∠GEF = ∠B, ∠GFE = ∠C, whence (*Cor.* 2. 32. 1.) ∠G = ∠A, and conſequently the Triangles ABC, GEF, are Equiangular.

I ſay, ∠𝔄 = ∠𝔇. ∠𝔅 = ∠𝔇𝔈𝔉. ∠ℭ = ∠𝔇𝔉𝔈.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AB:BC::GE:EF. 4. |
| Alſo | 2 | AB:BC::DE:EF. *Hyp.* |
| ∵ c. 1. 2. | 3 | GE:EF::DE:EF. 11. 5. |
| ∵ c. 3. | 4 | GE = DE. 9. 5. |
| Again | 5 | ∠GEF = ∠B. *Conſt.* |
| Alſo | 6 | ∠DEF = ∠B. *Hyp.* |
| ∵ c. 5. 6. | 7 | ∠GEF = ∠DEF. *Ax.* 1. |
| Laſtly | 8 | EF is common. |
| ∵ c. 4. 8. 7. | 9 | GE = DE, EF is common, ∠GEF = ∠DEF. |
| ∵ c. 9. | 10 | ∠G = ∠D, ∠GEF = ∠DEF, ∠GFE = ∠DFE. 4. 1. |
| Alſo | 11 | ∠G = ∠A, ∠GEF = ∠B, ∠GFE = ∠C. *Conſt.* |
| ∴ c. 11. 10. | 12 | ∠𝔄 = ∠𝔇. ∠𝔅 = ∠𝔇𝔈𝔉. ∠ℭ = ∠𝔇𝔉𝔈. *Ax.* 1. |

Q. E. D.

*The Seventh Propoſition is needleſs.*

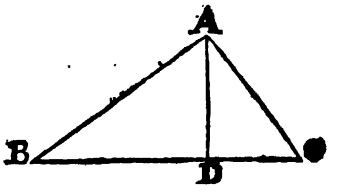
PROP.

## PROP. VIII.

*A Perpendicular* AD, *let fall from the Right-angle* BAC, *of a Right-angled Triangle* ABC, *upon the oppoſite Side* BC, *divides the Triangle* ABC, *into two others* ADB, ADC, *ſimilar to its ſelf.*

That is, △ˢ ABC, ADB, ADC, are Similar.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | ADB is a r∠. *Def.* 10. 1. |
| ∵ c. 1. | 2 | ∠BAD + ∠B = r∠. *Cor.* 3. 32. 1. |
| Alſo | 3 | BAC is a r∠. *Hyp.* |
| ∵ c. 3. | 4 | ∠C + ∠B = r∠. *Cor.* 3. 32. 1. |
| ∵ c. 2. 4. | 5 | ∠BAD + ∠B = ∠C + ∠B. *Ax.* 1. |
| 5 — ∠B. | 6 | ∠BAD = ∠C. *Ax.* 12. |
| Again | 7 | ADC is a r∠. *Def.* 10. 1. |
| ∵ c. 7. | 8 | ∠C + ∠CAD = r∠. *Cor.* 3. 32. 1. |
| ∵ c. 4. 8. | 9 | ∠C + ∠B = ∠C + ∠CAD. *Ax.* 1. |
| 9 — ∠C. | 10 | ∠B = ∠CAD. *Ax.* 12. |
| Laſtly c. 3. 1. 7. | 11 | ∠BAC = ∠ADB = ∠ADC. *Ax.* 21. |
| ∴ c. 11. 10. 6. | 12 | ∠BAC = ∠ADB = ∠ADC; ∠B = ∠CAD; ∠BAD = ∠C. |
| ∵ c. 12. | 13 | △ˢ ABC, ADB, ADC, are Similar. *Scho.* 1. 4. Q. E. D. |

## COROLLARIES.

1 In any Right angled Triangle, a Perpendicular being let fall from the Right-angle upon the Hypothenuſe, will (*Def.* 1.) be a mean Proportional between the Segments of the Hypothenuſe. That is, BD:AD::AD:DC.

2 The Baſe BA, is a mean Proportional, between the Hypothenuſe BC, and that Segment of the Hypothenuſe BD, next to the Baſe. That is, BC:BA::BA:BD.

3 The Cathetus AC, is a mean Proportional, between the Hypothenuſe BC, and that Segment of the Hypothenuſe DC, next to the Cathetus. That is, BC:AC::AC:DC.

## PROP. IX.

*From a Right-line given* AB, *to cut off any Part required, as one third* AG.

## PREPARATION.

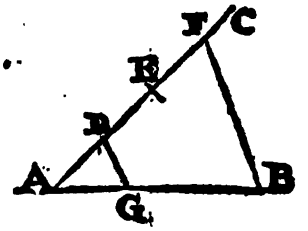
From the Point A, draw an infinite Line AC anywiſe, in which take (3. 1.) any three equal Parts AD, DE, EF; then join FB, to which from D, draw (31. 1.) the Parallel DG.

I ſay, $\mathfrak{AG} = \frac{1}{3}\mathfrak{AB}$.

DEMON-

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | FD:AD::BG:AG. 2. |
| ∴ c. 1. | 2 | FD + AD:AD::BG + AG:AG. 18. 5. |
| i. e. | 3 | AF:AD::AB:AG. |
| ∴ e. 3. | 4 | AD:AF::AG:AB. *Cor.* 7. 5. |
| But | 5 | AD = ½AF. *Const.* |
| ∴ c. 4. 5. | 6 | ΔG = ½ΔB. Q. E. F. & D. |



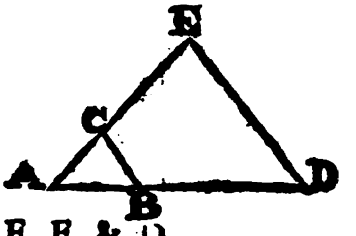

## PROP. X.

*To divide a given Line* AD, *in the same manner as another given Line* AE *is divided, so that* AC:CE::AB:BD.

## PREPARATION.

Join the two given Lines AD, AE, at any Angle DAE; then having joined ED; from the Point of Division C, draw (31. 1.) CB ‖ ED.

I say, ΔC:CE::ΔB:BD.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | CB ‖ ED. *Const.* |
| ∴ c. 1. | 2 | ΔC:CE::ΔB:BD. 2. Q. E. F. & D. |

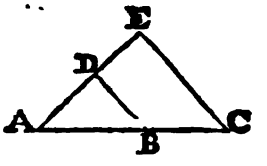
## PROP. XI.

*Two Right-lines* AB, AD, *being given, to find out a Third* DE, *in Proportion to them, so that* AB:AD :: AD:DE.

### PREPARATION.

Join BD, and from AB produced, take BC = AD. Thro' C, draw (31. 1.) CE ‖ BD; with which let AD produced meet in E.

I say, AB:AD::AD:DE.




### DEMONSTRATION.

1 AB:BC::AD:DE. 2.
And 2 BC = AD. *Const.*
∴ e. 1. 2. 3 AB:AD::AD:DE. *Sub.* Q. E. F. & D.

## PROP. XII.

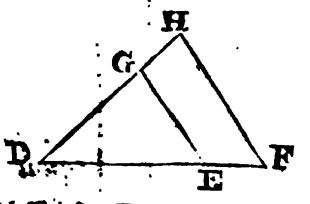
*Three Right-lines* DE, EF, DG, *being given, to find out a Fourth* GH, *in Proportion to them, so that* DE:EF :: DG:GH.

### PREPARATION.

Join EG, and thro' F, draw (31. 1.) FH ‖ EG; with which let DG produced meet in H.

I say,

I say, DC : CF :: DG : GH.




DEMONSTRATION.

1 FH ‖ EG. *Const.*
2 DC : CF :: DG : GH. 1. Q. E. F. & D.

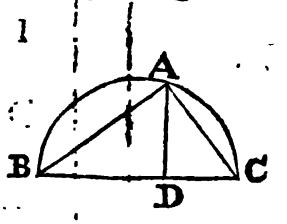
## PROP. XIII.

*Two Right-lines* BD, DC, *being given, to find out a mean Proportional* AD, *so that* BD : AD :: AD : DC.

## PREPARATION.

Upon the whole Line BC, as a Diameter, describe a Semicircle BAC, and having from D, erected a Perpendicular AD, meeting with the Periphery in A, draw AB, AC, so (31. 3.) shall BAC be a Right-angle.

I say, BD : AD :: AD : DC.




DEMONSTRATION.

1 BAC is a rt ∠. *Const.*
2 BD : AD :: AD : DC. *Cor.* 1. 8. Q. E. F. & D.

## COROLLARY.

Hence, a Right-line drawn in a Circle, from any Point of the Diameter perpendiculary, and extended to the Circumference, is a mean Proportional betwixt the two Segments of the Diameter.

## PROP. XIV.

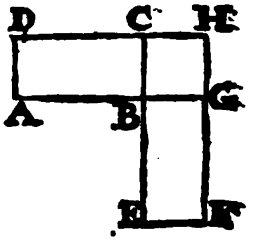
*Equal Parallelograms* BD, BF, *having one Angle* ABC, *equal to one* EBG, *have the Sides about the equal Angles reciprocal, ſo that* AB:BG::EB:BC. *And thoſe Parallelograms* BD, BF, *which have one Angle* ABC, *equal to one* EBG, *and the Sides which are about the equal Angles reciprocal, ſo that* AB:BG::BB:BC, *are equal.*

*Hyp.* 1 Suppoſe ▭BD = ▭BF, ∠ABC = ∠EBG.

### PREPARATION.

Let the Sides AB, BG, about the equal Angles, be ſet in a ſtraight Line, ſo (*Schol.* 15. 1.) ſhall EBC be likewiſe a ſtraight Line; alſo let DC, FG, be produced till they meet in H.

I ſay, 𝔄𝔅:𝔅𝔊::𝔈𝔅:𝔅𝔈.




### DEMONSTRATION.

1 AB:BG::▭BD:▭BH. 1.
And 2 ▭BD = ▭BF. *Hyp.*
∴ e. 1. 2. 3 AB:BG::▭BF:▭BH. *Sub.*
Alſo 4 EB:BC::▭BF:▭BH. 1.
∴ e. 3. 4. 5 𝔄𝔅:𝔅𝔊::𝔈𝔅:𝔅𝔈. 11. 5. Q. E. D.

*Hyp.* 2 Suppoſe ∠ABC = ∠EBG, AB:BG::EB:BC.

I ſay, ▭𝔄𝔅𝔇 = ▭𝔅𝔊𝔉.

DEMON.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AB:BG::EB:BC. *Hyp.* |
| Alſo | 2 | AB:BG::▭BD:▭BH. 1. |
| ∵ e. 1. 2. | 3 | EB:BC::▭BD:▭BH. 11. 5. |
| Alſo | 4 | EB:BC::▭BF:▭BH. 1. |
| ∵ e. 3. 4. | 5 | ▭BD:▭BH::▭BF:▭BH. 11. 5. |
| ∵ e. 5. | 6 | ▭𝔄𝔇 = ▭𝔄𝔉. 9. 5. Q. E. D. |

## PROP. XV.

*Equal Triangles* ACB, DEB, *having one Angle* ABC *equal to one* DBE, *have their Sides which are about the equal Angles reciprocal, ſo that* AB:BE::DB:BC. *And thoſe Triangles that have one Angle* ABC, *equal to one* DBE, *and the Sides which are about the equal Angles reciprocal, ſo that* AB:BE::DB:BC, *are equal.*

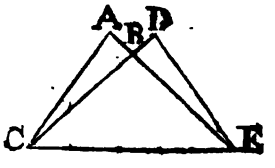
*Hyp.* 1 Suppoſe △ACB = △DEB, ∠ABC = ∠DBE.

## PREPARATION.

Let the Sides AB, BE, which are about the equal Angles, be ſet in a ſtraight Line, ſo (*Scho.* 15. 1.) ſhall DBC be likewiſe a ſtraight Line; alſo join CE.

I ſay, 𝔄𝔅:𝔅𝔈::𝔇𝔅:𝔅𝔈.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | △ACB = △DEB. *Hyp.* |
| ∵ e. 1. | 2 | △ACB:△ECB::△DEB:△ECB. 7. 5. |
| Alſo | 3 | △ACB:△ECB::AB:BE. 1. |
| ∵ e. 3. 2. | 4 | AB:BE::△DEB:△ECB. 11. 5. |
| Alſo | 5 | DB:BC::△DEB:△ECB. 1. |
| ∵ e. 4. 5. | 6 | 𝔄𝔅:𝔅𝔈::𝔇𝔅:𝔅𝔈. 11. 5. Q. E. D. |

| | | |
|---|---|---|
| *Hyp.* | 2 | Suppose $\angle ABC = \angle DBE$, $AB : BE :: DB : BC$. |
| | | I say, $\triangle \mathfrak{A}\mathfrak{C}\mathfrak{B} = \triangle \mathfrak{D}\mathfrak{C}\mathfrak{B}$. |

DEMONSTRATION.

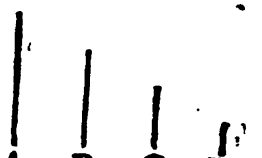
| | | |
|---|---|---|
| | 1 | $AB : BE :: DB : BC$. *Hyp.* |
| Alſo | 2 | $AB : BE :: \triangle ACB : \triangle ECB$. 1. |
| ∵ c. 1. 2. | 3 | $DB : BC :: \triangle ACB : \triangle ECB$. 11. 5. |
| Alſo | 4 | $DB : BC :: \triangle DEB : \triangle ECB$. 1. |
| ∵ c. 3. 4. | 5 | $\triangle ACB : \triangle ECB :: \triangle DEB : \triangle ECB$. 11. 5. |
| ∵ c. 5. | 6 | $\triangle \mathfrak{A}\mathfrak{C}\mathfrak{B} = \triangle \mathfrak{D}\mathfrak{C}\mathfrak{B}$. 9. 5. Q. E. D. |

## PROP. XVI.

*If four Lines* A, B, C, D, *are Proportional, that is, if* A : B :: C : D, *the Rectangle of the two Extreams* A, D, *is equal to the Rectangle of the two Means* B, C. *And if the Rectangle of the two Extreams* A, D, *be equal to that of the two Means* B, C, *the four Lines are Proportional.*

| | | |
|---|---|---|
| *Hyp.* | 1 | Suppoſe A : B :: C : D. |
| | | I ſay, $\mathfrak{A}\mathfrak{D} = \mathfrak{B}\mathfrak{C}$. |

DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | A : B :: C : D. *Hyp.* |
| ∵ c. 1. | 2 | $\mathfrak{A}\mathfrak{D} = \mathfrak{B}\mathfrak{C}$. *Lem.* 1. 5. Q. E. D. |
| *Hyp.* | 2 | Suppoſe $AD = BC$. |
| | | I ſay, $\mathfrak{A} : \mathfrak{B} :: \mathfrak{C} : \mathfrak{D}$. |

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $AD = BC$. *Hyp.* |
| ∵ c. 1. | 2 | $\mathfrak{A} : \mathfrak{B} :: \mathfrak{C} : \mathfrak{D}$. *Lem.* 2. 5. Q. E. D. |

PROP.

## PROP. XVII.

*If three Lines* A, B, C, *be continually Proportional, that is, if* A:B::B:C, *the Rectangle of the two Extreams* A, C, *is equal to the Square of the Mean* B. *And if the Rectangle of the two Extreams* A, C, *be equal to the Square of the Mean* B, *the three Lines are Proportional.*

*Hyp.* 1 Suppose A:B::B:C.

### PREPARATION.

Make D = B.

I say, $\mathfrak{A}\mathfrak{C} = \mathfrak{B}\mathfrak{q}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | A:B::B:C. *Hyp.* |
| And | 2 | D = B. *Const.* |
| ∴ e. 1. 2. | 3 | A:B::D:C. *Sub.* |
| ∴ e. 3. | 4 | AC = BD. 16. |
| ∴ e. 2. 4. | 5 | AC = BB. *Sub.* |
| i. e. | 6 | $\mathfrak{A}\mathfrak{C} = \mathfrak{B}\mathfrak{q}$. Q. E. D. |

*Hyp.* 2 Suppose AC = *Bq.*

I say, $\mathfrak{A}:\mathfrak{B}::\mathfrak{B}:\mathfrak{C}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AC = *Bq.* *Hyp.* |
| i. e. | 2 | AC = BB. |
| Also | 3 | D = B. *Const.* |
| ∴ e. 2. 3. | 4 | AC = BD. *Sub.* |
| ∴ e. 4. | 5 | A:B::D:C. 16. |
| ∴ e. 3. 5. | 6 | $\mathfrak{A}:\mathfrak{B}::\mathfrak{B}:\mathfrak{C}$, *Sub.* Q. E. D. |

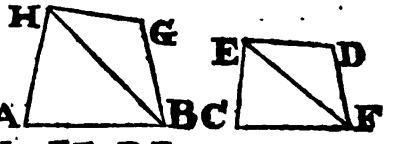
## PROP. XVIII.

*Upon a Right-line given* AB, *to describe a Right-lined Figure* ABGH, *similar to a Right-lined Figure given* CFDE.

## PREPARATION.

Make (23. 1.) ∠ABH=∠CFE, ∠GBH=∠DFE, ∠BHA=∠FEC, ∠BHG=∠FED, so (*Cor.* 2. 32. 1.) is ∠A=∠C, ∠G=∠D, and consequently the △'AB H, CFE, as also the △'GBH, DFE, are Equiangular; also (*Ax.* 11.) ∠ABH+∠GBH=∠CFE+∠DFE; ∠BHA+∠BHG=∠FEC+∠FED; that is, ∠A BG=∠CFD, ∠AHG=∠CED.

I say, ▱ABGH, ▱CFDE, are Similar.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AH:BH::CE:FE; BH:GH::FE:DE. 4. |
| ∴ e. 1. | 2 | AH:GH::CE:DE. 22. 5. |
| Also | 3 | AB:BH::CF:FE; BH:BG::FE:FD. 4. |
| ∴ e. 3. | 4 | AB:BG::CF:FD. 22. 5. |
| Again | 5 | AB:AH::CF:CE; BG:GH::FD:DE. 4. |
| Also | 6 | ∠A=∠C, ∠G=∠D, ∠ABG=∠CFD, ∠AHG=∠CED. *Const.* |
| ∴ e. 6. 5. 4. 2. | 7 | ∠A=∠C, AB:AH::CF:CE; ∠G=∠D, BG:GH::FD:DE; ∠ABG=∠CFD, AB:BG::CF:FD; ∠AHG=∠CED, AH:GH::CE:DE. |
| ∴ e. 7. | 8 | ▱ABGH, ▱CFDE, are Similar. *Def.* 1. Q.E.F. & D. |

LEMMA

## LEMMA I.

*If three Quantities* A, AR, ARR, *be continually Proportional, that is, if* A:AR::AR:ARR; *then as the First is to the Third, so is the Square of the First to the Square of the Second.*

That is, $\mathfrak{A}:\mathfrak{A}\mathfrak{R}\mathfrak{R}::\mathfrak{A}\mathfrak{A}:\mathfrak{A}\mathfrak{A}\mathfrak{R}\mathfrak{R}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | A×ARR = AR×AR. |
| ∵ e. 1. | 2 | A:AR::AR:ARR. *Lem.* 2. 5. |
| Alſo | 3 | A×AARR = ARR×AA. |
| ∵ e. 3. | 4 | $\mathfrak{A}:\mathfrak{A}\mathfrak{R}\mathfrak{R}::\mathfrak{A}\mathfrak{A}:\mathfrak{A}\mathfrak{A}\mathfrak{R}\mathfrak{R}$. *Lem.* 2. 5. Q. E. D. |

## PROP. XIX.

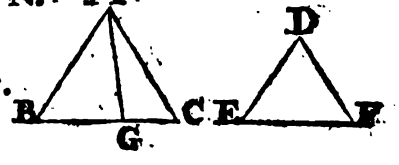
*Similar Triangles* ABC, DEF, *are in the Duplicate Ratio, or as the Squares of their Homologous Sides* BC, EF.

### PREPARATION.

To the Sides BC, EF, find (11.) a third Proportional BG, ſo that BC:EF::EF:BG, and join AG.

I ſay, $\triangle\mathfrak{A}\mathfrak{B}\mathfrak{C}:\triangle\mathfrak{D}\mathfrak{E}\mathfrak{F}::\mathfrak{B}\mathfrak{C}\mathfrak{q}:\mathfrak{E}\mathfrak{F}\mathfrak{q}$.

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AB:BC::DE:EF. *Def.* 1. |
| ∵ e. 1. | 2 | BC:EF::AB:DE. *Cor.* 7. 5. |
| Alſo | 3 | BC:EF::EF:BG. *Conſt.* |
| ∵ e. 2. 3. | 4 | AB:DE::EF:BG. 11. 5. |
| Again | 5 | ∠B = ∠E. *Def.* 1. |
| ∵ e. 5. 4. | 6 | ∠B = ∠E, AB:DE::EF:BG. |
| ∵ e. 6. | 7 | △ABG = △DEF. 15. |
| Laſtly | 8 | BC:BG::△ABC:△ABG. 1. |
| ∵ e. 7. 8. | 9 | BC:BG::△ABC:△DEF. *Sub.* |
| Alſo e. 3. | 10 | BC:BG::BCq:EFq. *Lem.* 1. |
| ∵ e 9. 10. | 11 | $\triangle\mathfrak{A}\mathfrak{B}\mathfrak{C}:\triangle\mathfrak{D}\mathfrak{E}\mathfrak{F}::\mathfrak{B}\mathfrak{C}\mathfrak{q}:\mathfrak{E}\mathfrak{F}\mathfrak{q}$. 11. 5. Q. E. D. |

## COROLLARIES.

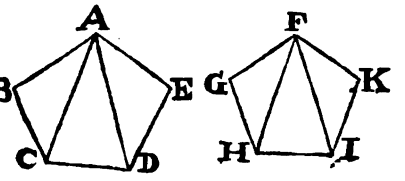
1 Becauſe (*Step.* 9.) BC; BG :: △ ABC: △ DEF, therefore if three Lines BC, EF, BG, be Proportional, then as the firſt BC, is to the third BG, ſo is the Triangle deſcribed upon the firſt BC, to the Similar Triangle deſcribed upon the ſecond EF.

2 Becauſe Equiangular Triangles are likewiſe (*Scho.* 1. 4.) Similar, therefore Equiangular Triangles, are alſo as the Squares of their Homologous Sides.

## PROP. XX.

1ſt *Similar Polygons* ABCDE, FGHIK, *are divided into an equal Number of Similar Triangles,* ABC *and* FGH, ACD *and* FHI, ADE *and* FIK. 2*d. Thoſe Similar Triangles are Homologous to the wholes* ABCDE, FGHIK, *ſo that* △ABC: △FGH::ABCDE:FGHIK::△ACD: △FHI:: △ADE: △FIK. 3*d. The Polygons* ABCDE, FGHIK, *are in the Duplicate Ratio, or as the Squares of their Homologous Sides* CD, HI.

That is, 1ſt **△ſ ABC and FGH, ACD and FHI, ADE and FIK, are Similar.**




DEMONSTRATION.

1 ∠B = ∠G, BA : BC :: GF : GH. *Def.* 1.
∵ e. 1. 2 △ABC, FGH, are Equiangular. 6.
∵ e. 2. 3 CA : BC :: HF : GH. 4.
Alſo e. 2. 4 ∠ACB = ∠FHG.
Again 5 ∠ACB + ∠ACD = ∠FHG + ∠FHI. *Def.* 1.
5 — 4 6 ∠ACD = ∠FHI. *Ax.* 12.
Moreover 7 BC : CD :: GH : HI. *Def.* 1.
∵ e. 3. 7. 8 CA : BC :: HF : GH; BC : CD :: GH : HI.

CA

| | | |
|---|---|---|
| ∴ e. 8. | 9 | CA:CD::HF:HI. 22. 5. |
| ∴ e. 6. 9. | 10 | ∠ACD = ∠FHI, CA:CD::HF:HI. |
| ∴ e. 10. | 11 | △ACD, FHI, are Equiangular. 6. |
| Laſtly | 12 | ∠E = ∠K, EA:ED::KF:KI. *Def.* 1. |
| ∴ e. 12. | 13 | △ADE, FIK, are Equiangular. 6. |
| ∴ e. 2. 11. 13. | 14 | △ABC and FGH, ACD and FHI, ADE and FIK, are Equiangular. |
| ∴ e. 14. | 15 | **△ABC and FGH, ACD and FHI, ADE and FIK, are Similar.** *Scho.* 1. 4. Q. E. D. |
| | 2d. | **△ABC:△FGH::ABCDE:FGHIK::△ACD:△FHI::△ADE:△FIK.** |

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | △, ABC and FGH, ACD and FHI, are Similar. *1ſt.* |
| ∴ e. 1. | 2 | △ABC:△FGH::GA*q*:HF*q*::△ACD:△FHI. 19. |
| ∴ e. 2. | 3 | △ABC:△FGH::△ACD:△FHI. 11. 5. |
| Again | 4 | △ACD and FHI, ADE and FIK, are Similar. *1ſt.* |
| ∴ e. 4. | 5 | △ACD:△FHI::DA*q*:IF*q*::△ADE:△FIK. 1. |
| ∴ e. 5. | 6 | △ACD:△FHI::△ADE:△FIK. 11. 5. |
| ∴ e. 3. 6. | 7 | △ABC:△FGH::△ACD:△FHI::ADE:△FIK. |
| ∴ e. 7. | 8 | △ABC:△FGH::△ABC+△ACD+△ADE △FGH+△FHI+△FIK. 12. 5. |
| i. e. | 9 | △ABC:△FGH::ABCDE:FGHIK. |
| ∴ e. 9. 7. | 10 | **△ABC:△FGH::ABCDE:FGHIK::△ACD:△FHI::△ADE:△FIK.** 11. 5. Q. E. D. |
| | 3d. | **ABCDE:FGHIK::CDq:HIq.** |

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | ABCDE:FGHIK::△ACD:△FHI. 2d. |
| And | 2 | CD*q*:HI*q*::△ACD:△FHI. 19. |
| ∴ e. 1. 2. | 3 | **ABCDE:FGHIK::CDq:HIq.** 11. 5. Q.E.D. |

## COROLLARIES.

1 Similar Quadrilateral Figures A C D E, F H I K, are alſo in the duplicate Ratio, or as the Squares of their homologous Sides C D, H I.

For 1 △A C D : △F H I :: △A D E : △F I K. *2d.*
i. e. 1. 2 △A C D : △F H I :: △A C D + △A D E : △F H I + △F I K. 12. 5.
i. e. 3 △A C D : △F H I :: A C D E : F H I K.
Alſo 4 △A C D : △F H I :: C D *q* : H I *q*. 19.
i. e. 3. 4. 5 A C D E : F H I K :: C D q : H I q. 11. 5.

2 Therefore (19.) Similar Triangles, (*Cor.* 1.) Similar Quadrilateral Figures, (20.) Similar Polygons, and conſequently univerſally, Similar Right-lined Figures, are in the duplicate Ratio, or as the Squares of their Homologous Sides.

3 If to C D, H I, be taken a third Proportional X, then as the firſt C D, is to the third X, ſo is the Quadrilateral Figure A C D E, deſcribed upon the firſt C D, to the Similar Quadrilateral Figure F H I K, deſcribed upon the ſecond H I, ſo is the Polygon A B C D E, deſcribed upon the firſt C D, to the Similar Polygan F G H I K, deſcribed upon the ſecond H I.

For 1 C D : X :: C D *q* : H I *q*. *Lem.* 1.
Alſo 2 A C D E : F H I K :: (*Cor.* 1.) C D *q* : H I *q* :: (*3d.*) A B C D E : F G H I K.
i. e. 1. 2. 3 C D : X :: A C D E : F H I K :: A B C D E : F G H I K. 11. 5.

There-

4 Therefore, if three Right-lines are Proportional, then as the Firſt is to the Third, ſo (*Cor.* 1. 19.) is the Triangle, ſo (*Cor.* 3.) is the Quadrilateral Figure, and Polygon, and conſequently ſo univerſally is the Right-lined Figure, deſcribed upon the Firſt. to the Similar One deſcribed upon the Second.

## PROP. XXI.

*Right-lined Figures* ABC, DEI, *which are Similar to the ſame Right-lined Figure* HGF, *are alſo Similar to one another.*

That is, **ABC, DEI, are Similar.**

### DEMONSTRATION.

$\angle A = \angle H = \angle D$, AB : AC :: HG : HF :: DE : DI;
1 $\angle C = \angle F = \angle I$, AC : CB :: HF : FG :: DI : IE; $\angle B = \angle G = \angle E$, AB : CB :: HG : FG :: DE : IE. *Def.* 1.
∵ e. 1. 2 ABC, HGF, DEI, are Similar. *Def.* 1.
∵ e. 2. 3 **ABC, DEI, are Similar.** Q. E. D.

## PROP. XXII.

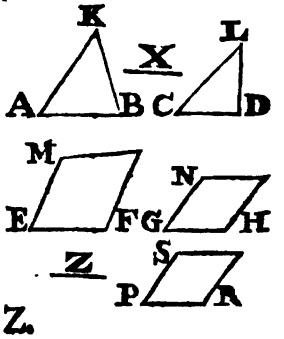
*If four Right-lines* AB, CD, EF, GH, *are Proportional, that is, if* AB:CD::EF:GH, *the Similar Right-lined Figures* KAB, LCD, MF, NH, *deſcribed upon thoſe Lines, are Proportional. And if the Similar Right-lined Figures* KAB, LCD, MF, NH, *are Proportional, that is, if* KAB:LCD::MF:NH, *the four Lines* AB, CD, EF, GH, *upon which they are deſcribed, are Proportional.*

*Hyp.* 1 Suppoſe AB:CD::EF:GH.

## PREPARATION.

To AB, CD, find (11.) a third Proportional X, ſo that AB:CD::CD:X; alſo to EF, GH, find (11.) a third Proportional Z, ſo that EF:GH::GH:Z.

I ſay, 𝔎𝔄𝔅:𝔏𝔇𝔅::𝔐𝔉:𝔑𝔊.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AB:CD::EF:GH. *Hyp.* |
| Alſo | 2 | AB:CD::CD:X. *Conſt.* |
| ∴ c. 1. 2. | 3 | EF:GH::CD:X. 11. 5. |
| Alſo | 4 | EF:GH::GH:Z. *Conſt.* |
| ∴ c. 3. 4. | 5 | CD:X::GH:Z. 11. 5. |
| ∴ c. 1. 5. | 6 | AB:CD::EF:GH; CD:X::GH:Z. |
| ∴ c. 6. | 7 | AB:X::EF:Z. 22. 5. |
| Alſo | 8 | AB:X::KAB:LCD. *Cor.* 4. 20. |
| ∴ c. 7. 8. | 9 | EF:Z::KAB:LCD. 11. 5. |
| Alſo | 10 | EF:Z::MF:NH. *Cor.* 4. 20. |
| ∴ c. 9. 10. | 11 | 𝔎𝔄𝔅:𝔏𝔇𝔅::𝔐𝔉:𝔑𝔊. 11. 5. Q. E. D. |

*Hyp.* 2 Suppoſe KAB:LCD::MF:NH.

PRE-

## PREPARATION.

To AB, CD, EF, find (12.) a fourth Proportional PR, so that AB:CD::EF:PR; then upon PR describe (18.) the Right-lined Figure SR, Similar to the Similar Right-lined Figures MF, NH.

I say, $\mathfrak{AB}:\mathfrak{CD}::\mathfrak{EF}:\mathfrak{GH}$.

## DEMONSTRATION.

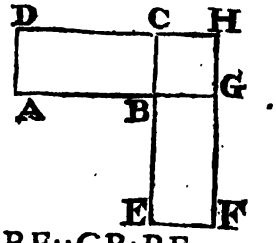
| | | |
|---|---|---|
| | 1 | AB:CD::EF:PR. *Const.* |
| ∵ e. 1. | 2 | KAB:LCD::MF:SR. *1st.* |
| Also | 3 | KAB:LCD::MF:NH. *Hyp.* |
| ∵ e. 2. 3. | 4 | MF:SR::MF:NH. 11. 5. |
| ∵ e. 4. | 5 | SR = to its Similar NH. 9. 5. |
| ∵ e. 5. | 6 | PR = GH. |
| ∵ e. 1. 6. | 7 | $\mathfrak{AB}:\mathfrak{CD}::\mathfrak{EF}:\mathfrak{GH}$. *Sub.* Q. E. D. |

## PROP. XXIII.

*Equiangular Parallelograms* BD, BF, *are in a Ratio compounded of that their Sides.*

## PREPARATION.

Let the Sides AB, BG, about the equal Angles, be set in a straight Line, so (*Scho.* 15. 1.) shall EBC, be likewise a straight Line; also let DC, FG, be produced 'till they meet in H.




I say, $\frac{\square\mathfrak{BD}}{\square\mathfrak{BF}} = \frac{\mathfrak{AB}}{\mathfrak{BG}} \times \frac{\mathfrak{CB}}{\mathfrak{BE}}$.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\square$BD:$\square$BH::AB:BG; $\square$BH:$\square$BF::CB:BE. 1. |
| ∵ e. 1. | 2 | $\frac{\square BD}{\square BH} = \frac{AB}{BG}$; $\frac{\square BH}{\square BF} = \frac{CB}{BE}$. *Def.* 2. 5. |
| Again | 3 | $\frac{\square BD}{\square BF} = \frac{\square BD}{\square BH} \times \frac{\square BH}{\square BF}$. *Def.* 5. |
| ∵ e. 2. 3. | 4 | $\frac{\square\mathfrak{BD}}{\square\mathfrak{BF}} = \frac{\mathfrak{AB}}{\mathfrak{BG}} \times \frac{\mathfrak{CB}}{\mathfrak{BE}}$. *Sub.* Q. E. D. |

## PROP. XXIV.

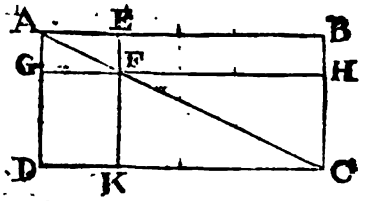
*In every Parallelogram* B D, *the Parallelograms* E G, K H, *which are about the Diameter* A C, *are Similar to the whole* B D, *and alſo to one another.*

## PREPARATION.

With the Right-lines GH, EK, drawn thro' the Point F, taken at Pleaſure in the Diagonal A C, parallel to the oppoſite Sides, divide the Parallelogram B D, into four Parallelograms E G, E H, G K, K H, ſo (*Scho.* 2. 4.) are the two Triangles A E F, F H C, Similar to the Triangle A B C, the two A G F, F K C, Similar to the Triangle A D C, and conſequently (21.) the three Triangles A E F, A B C, F H G, as alſo the three A G F, A D C, F K C, are Similar.

I ſay, ☐ E G, K H, B D, are Similar.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | ∠GFE = ∠GAE. 34. 1. |
| Alſo | 2 | ∠GFE = ∠KFH. 15. 1. |
| ∵ e. 1. 2. | 3 | ∠GAE = ∠KFH. *Ax.* 1. |
| Alſo | 4 | ∠KCH = ∠KFH. 34. 1. |
| ∵ e. 2. 4. | 5 | ∠GFE = ∠KCH. *Ax.* 1. |
| Again | 6 | ∠AEF = ∠ABC = ∠FHC; ∠AGF = ∠ADC = ∠FKC. 29. 1. |
| ∵ e. 6. | 7 | AE:EF::AB:BC::FH:HC; AG:GF::AD:DC::FK:KC. *Def.* 1. |
| Now | 8 | AE = GF, AB = DC, FH = KC. 34. 1. |
| ∵ e. 7. 8. | 9 | GF:EF::DC:BC::KC:HC; AG:AE::AD:AB::FK:FH. *Sub.* |

∠GFE

∠GFE = ∠KCH, GF:EF::DC:BC::KC:HC; ∠GAE = ∠KFH, AG:AE::AD:AB::FK:FH, ∠AEF = ∠ABC = ∠FHC, AE:EF::AB:BC::FH:HC; ∠AGF = ∠ADC = ∠FKC, AG:GF::AD:DC::FK:KC.

∴ e. 5. 9. 3. 9. 6. 7. 10

∴ e. 10. 11 ▭CG, KH, BD, are Similar. *Def.* 1. Q. E. D.

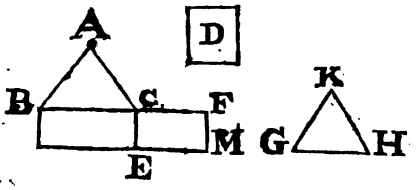
## PROP. XXV.

*Unto a Right-lined Figure given* ABC, *to describe another* KGH *Similar, which shall also be equal to another Right-lined Figure given* D.

## PREPARATION.

Upon BC, make (44. 1.) ▭BE = ABC, and upon CE, make (45. 1.) ▭CM = D; betwixt BC and CF, find (13.) a mean Proportional GH, so that BC:GH::GH:CF, upon which make (18.) KGH Similar to ABC.

I say, KGH = D.




## DEMONSTRATION.

1 BC:CF::ABC:KGH. *Cor.* 4. 20.

Also 2 BC:CF::▭BE:▭CM. 1.

∴ e. 1. 2. 3 ABC:KGH::▭BE:▭CM. 11. 5.

Again 4 ▭BE = ABC. *Const.*

∴ e. 3. 4 5 ABC:KGH::ABC:▭CM. *Sub.*

∴ e. 5. 6 KGH = ▭CM. 9. 5.

Also 7 D = ▭CM. *Const.*

∴ e. 6. 7. 8 KGH = D. *Ax.* 1. Q. E. F. & D.

*We shall here omit the 26th, 27th, 28th, and 29th Propositions, that are but of little Consequence.*

## PROP. XXX.

*To cut a finite Right-line given* A B, *according to extream and mean Ratio, so that* AB : AG :: AG : GB.

### PREPARATION.

Make (11. 2.) $AG^q = AB \times GB$.

I say, AB : AG :: AG : GB.

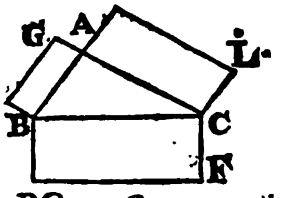
### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $AG^q = AB \times GB$. *Const.* |
| ∴ e. 1. | 2 | AB : AG :: AG : GB. 17. Q. E. F. & D. |

## PROP. XXXI.

*In any Right-angled Triangle* A B C, *the Sum of any two Similar Right-lined Figures* B G, A L, *described upon the Sides* A B, A C, *is equal to the Similar Right-lined Figure* B F, *described upon the Side* B C, *subtending the Right-angle* B A C.

That is, BG + AL = BF.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $BG : BF :: AB^q : BC^q$; $AL : BF :: AC^q : BC^q$. *Cor.* 2. 30. |
| ∴ e. 1. | 2 | $BG + AL : BF :: AB^q + AC^q : BC^q$. 24. 5. |
| ∴ e. 2. | 3 | $BG + AL : AB^q + AC^q :: BF : BC^q$. 16. 5. |
| ∴ e. 3. | 4 | $AB^q + AC^q : BG + AL :: BC^q : BF$. *Cor.* 7. 5. |
| Now | 5 | $AB^q + AC^q = BC^q$. 47. 1. |
| ∴ e. 4. 5. | 6 | BG + AL = BF. 14. 5. Q. E. D. |

*The* 32*d*, *Proposition is useless.*

PROP.

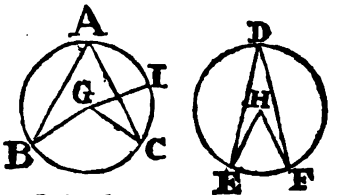
## PROP. XXXIII.

*In equal Circles* ABC, DEF, *the Angles have the same Ratio with their Arches* BC, EF, *on which they insist; whether they be at the Centers, as* BGC, EHF, *or at the Circumferences, as* A, D.

## PREPARATION.

Set off (the Circles being equal) Arch CI = Arch EF, and draw GI, so (27. 3.) is ∠CGI = ∠EHF.

I say, Arch BC : Arch EF :: ∠BGC : ∠EHF :: ∠A : ∠D.




## DEMONSTRATION.

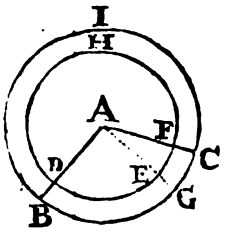
| | | |
|---|---|---|
| | 1 | As the Measure of the ∠BGC : the Measure of the ∠CGI :: ∠BGC : ∠CGI. |
| But | 2 | Arch BC is the Measure of the ∠BGC, Arch CI the Measure of the ∠CGI. |
| ∵ e. 1. 2. | 3 | Arch BC : Arch CI :: ∠BGC : ∠CGI. *Sub.* |
| Again | 4 | Arch CI = Arch EF, ∠CGI = ∠EHF. *Const.* |
| ∵ e. 3. 4. | 5 | Arch BC : Arch EF :: ∠BGC : ∠EHF. *Sub.* |
| Lastly | 6 | ∠BGC : ½∠BGC :: ∠EHF : ½∠EHF. 15. 5. |
| ∵ e. 6. | 7 | ∠BGC : ∠EHF :: ½∠BGC : ½∠EHF. 16. 5. |
| But | 8 | ∠A = ½∠BGC, ∠D = ½∠EHF. 20. 3. |
| ∴ e. 7. 8. | 9 | ∠BGC : ∠EHF :: ∠A : ∠D. *Sub.* |
| ∵ e. 5. 9. | 10 | Arch BC : Arch EF :: ∠BGC : ∠EHF :: ∠A : ∠D. 11. 5. Q. E. D. |

## SCHOLIUM.

In the same Circle ABC, the Angles BGC, CGI, at the Center, (and consequently their Halves at the Circumference) have (*Step.* 3.) the same Ratio with their Arches BC, CI. The same is true of the Sectors GBCG, GCIG; as also of the Sectors GBCG, HEFH, in the same, and equal Circles, considering them as Angles. CO-

## COROLLARIES.

1 Becauſe as the Angle B A C, is to a Right-angle B A G, ſo *(Scho. 33.)* is the Arch B C to the Quadrant B G. Therefore as the Angle B A C, at the Center, is to four Right-angles, ſo is the Arch B C, on which it inſiſts, to four Quadrants, that is, to the whole Periphery




2 The Arches D F, B C, of unequal Circles, which ſubtend equal Angles, whether at the Centers, as D A F, B A C, or at the Peripheries, are Similar; ſo that as the Arch D F, is to the whole Periphery D F H, ſo is the Arch B C, to the whole Periphery B C I.

For 1 ∠ D A F : 4r ∠ :: Arch D F : D F H. *Cor.* 1.
i. e. 2 ∠ B A C : 4r ∠ :: Arch D F : D F H.
Alſo 3 ∠ B A C : 4r ∠ :: Arch B C : B C I. *Cor.* 1.
∴ e. 2. 3. 4 Arch D F : D F H :: Arch B C : B C I. 11. 5.

3 Two Semidiameters A B, A C, cut off *(Cor. 2.)* Similar Arches D F, B C, from Concentric Peripheries.

*The End of the Sixth Book.*

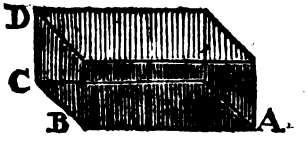
# EUCLID's ELEMENTS,

Demonſtrated after a New, Plain, and Eaſie METHOD.

## BOOK XI.

### DEFINITIONS.

1 A **Body** or **Solid**, is that which hath Length, Breadth, and Thickneſs.



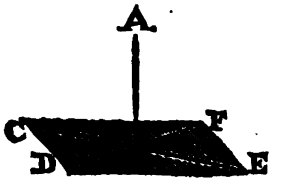

As the Figure A D, whoſe Length is AB, Breadth BC, and Thickneſs CD.

Becauſe a **Body** or **Solid**, hath Length, Breadth, and Thickneſs, it is called a Quantity of three Dimenſions, and is ſuppoſed to conſiſt of an Infinite Number of Surfaces.

2 The **Extremities**, or **Ends** of a Solid, are Superficies's or Surfaces.

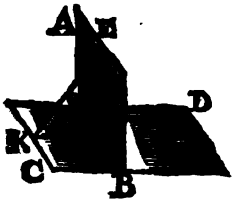
3 **A Right-line is Perpendicular to a Plane,** when 'tis Perpendicular to, or makes Right-angles with, all the Right-lines it meets in that Plane.

Thus the Right-line A B, is Perpendicular to the Plane C E, if it be Perpendicular to, or make Right-angles with, all the Right-lines B C, B D, B E, B F, it meets in that Plane.




4 **One Plane is Perpendicular to another,** when a Perpendicular Line drawn upon one of them to the common Section, is alſo Perpendicular to the other.

Thus the Plane A B, is Perpendicular to the Plane C D, becauſe the Line E F, drawn upon one of the Planes A B, Perpendicular to the common Section G B, is alſo Perpendicular to the other C D.

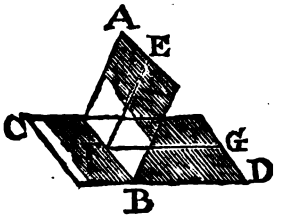



5 If the Line E K, be not Perpendicular to the Plane C D, and from the Point E, a Perpendicular E F, be drawn to it, and alſo the Line K F; the Angle E K F, is the **Inclination of the Line E K to the Plane C D.**

6 The **Inclination of one Plane to another,** is the Acute Angle form'd by the two Perpendiculars, drawn upon each Plane, to their common Section.

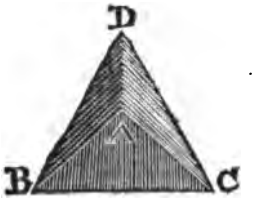
Thus

Thus the **Inclination** of the Plane A B, to the Plane C D, is the Acute Angle E F G, form'd by the two Perpendiculars E F, G F, drawn upon each Plane to the common Section.

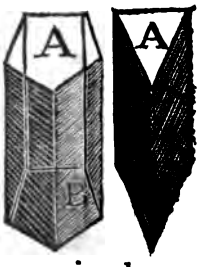



7 **Parallel Planes**, are such as being produced infinitely, will never meet.

8 A **Solid Angle** A, is that which is contained under more than two Plane Angles B A C, B A D, C A D, not being in the same Plane, but meeting all at one Point A.




9 A **Prism**, is a Solid Figure contained under Planes, whereof the two opposite A, B, are Parallel, Similar, and Equal; but the others are Parallelograms.




10 A **Parallelepipedon**, is a Solid Figure, contained under six Quadrilateral Figures, whereof those which are opposite are Parallel.

11 **Similar Solids**, are such as are bounded by an equal Number of Similar Planes.

12 **Similar and equal Solids**, are such as are bounded by an equal Number of Similar and equal Planes.

## PROP. I.

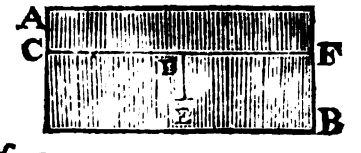
*A Right-line* CD, *in a Plane* AB, *if produced, will ſtill lie in that Plane.*

## PREPARATION.

Draw (11. 1.) from the Point D, in the Plane AB, the Right-line DE ⊥ CD, and another DF ⊥ DE.

I ſay, **The Right-line CD produced, is in the Plane AB.**

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | ∠EDC + ∠EDF = 2 r∠s. *Def.* 10. 1. |
| ∵ e. 1. | 2 | The Right-lines CD, DF, make one Right-line CF. 14. 1. |
| But | 3 | The Right-lines CD, DF, are in the Plane AB. *Conſt.* |
| ∴ e. 2. 3. | 4 | The Right-line CF, is in the Plane AB. |
| i. e. | 5 | **The Right-line CD produced, is in the Plane AB.** |

Q. E. D.

## PROP. II.

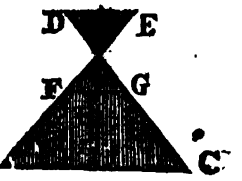
*Every Triangle* ABC, *is in the ſame Plane; as are alſo two Right-lines* CD, BE, *which cut each other.*

## PREPARATION.

Draw at Pleaſure in the △ABC, the Right-line FG.

I ſay

I say, △ABC is in the same Plane, as are also CD, BE.




DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | If part AFG, of the △ABC, should be in one Plane, part BCGF in another. |
| Then | 2 | AF, FB, would be in different Planes. |
| But | 3 | AF, FB, are in the same Plane. 1. |
| ∴ | 4 | Part of the △ABC, is not in one Plane, part in another. |
| ∴ | 5 | △ABC is in the same Plane. |
| ∴ e. 5. | 6 | CA, BA, are in the same Plane. |
| ∴ e. 6. | 7 | CD, BE, are in the same Plane. 1. |
| ∴ e. 5. 7. | 8 | △ABC is in the same Plane, as are also CD, BE. Q. E. D. |

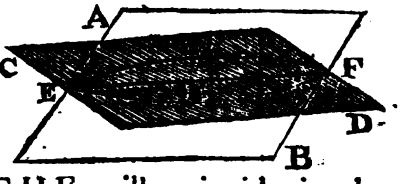
## PROP. III.

*The common Section* EF, *of two Planes* AB, CD, *is a Right-line.*

## PREPARATION.

Thro' the Points E, F, of the common Section EF, draw (*post.* 1.) in the Plane AB, the Right-line EGF, in Plane CD, the Right-line EHF.

I say, The common Section EF is a Right-line.




DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | Either the Right-lines EGF, EHF, will coincide in the common Section, and make one Right-line EF. |
| Or | 2 | Two Right-lines EGF, EHF, will include Space. |
| But | 3 | Two Right-lines cannot include Space. *Ax.* 20. |
| ∴ | 4 | The Right-lines EGF, EHF, will coincide in the common Section, and make one Right-line EF. |
| ∴ | 5 | The common Section EF is a Right-Line. Q. E. D. |

## PROP. IV.

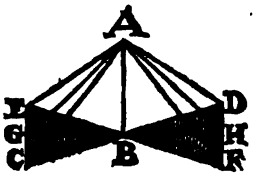
*If a Right-line* AB, *be perpendicular to two other* CD, EF, *which cut each other in a Point* B, *ſo that* ABC, ABD, ABE, ABF, *be Right-angles, it will be alſo Perpendicular to the Plane of the ſame Lines.*

## PREPARATION.

Make BC = BD = BE = BF, and having joined EC, DF, and drawn thro' the Point B, the Line GH at Pleaſure, join AE, AG, AC, AF, AH, AD.

I ſay, AB ⊥ the Plane ECFD.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | BC = BD = BE = BF, AB is common, ∠ABC = ∠ABD = ∠ABE = ∠ABF. *Conſt.* |
| ∵ e. 1. | 2 | AC = AD, AE = AF. 4. 1. |
| Again | 3 | ∠CBE = ∠DBF, ∠CBG = ∠DBH. 15. 1. |
| ∵ e. 1. 3 | 4 | BC = BD, BE = BF, ∠CBE = ∠DBF. |
| ∵ e. 4. | 5 | EC = DF, ∠BCG = ∠BDH. 4. 1. |
| ∵ e. 4. 5. 3. | 6 | BC = BD, ∠BCG = ∠BDH, ∠CBG = ∠DBH. |
| ∵ e. 6. | 7 | CG = DH, BG = BH. 26. 1. |
| Laſtly e. 2. 5. | 8 | AC = AD, AE = AF, EC = DF. |
| ∵ e. 8. | 9 | ∠ACG = ∠ADH. 8. 1. |
| ∵ e. 7. 8. 9. | 10 | CG = DH, AC = AD, ∠ACG = ∠ADH. |
| ∵ e. 10. | 11 | AG = AH. 4. 1. |
| ∵ e. 7. 11. 1. | 12 | BG = BH, AG = AH, AB is common. |
| ∵ e. 12. | 13 | ∠ABG = ∠ABH. 8. 1. |
| ∵ e. 13 | 14 | AB ⊥ GH. *Def.* 10. 1. |
| Alſo | 15 | After the ſame Manner, is AB ⊥ all the Right-lines it meets in the Plane ECFD. |
| ∵ | 16 | AB ⊥ the Plane ECFD. *Def.* 3. Q. E. D. |

PROP.

## PROP. V.

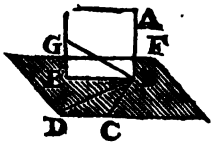
*If a Right-line* AB, *be perpendicular to three others* BC, BD, BE, *which cut each other at the ſame Point* B, *they will be all three in the ſame Plane.*

## PREPARATION.

If BE be ſaid not to be the common Section of the Planes AE, FD, and conſequently not to be in the ſame Plane with BC, BD, which (2.) are in the ſame Plane FD, let any other Right-line BG, be the common Section; ſo (AB being *(Hyp.)* ⊥ BC, BD, and conſequently (4.) to the Plane FD) will ∠ABG, form'd by ⊥ AB and that Right-line BG, be (*Def.* 3.) Right.

I ſay, BC, BD, BE, are in the ſame Plane

## DEMONSTRATION.




| | | |
|---|---|---|
| For | 1 | If any other Line, beſides BE, ſhould be the common Section of the two Planes; as BG. |
| Then | 2 | ∠ABG, form'd by ⊥ AB and that Right-line BG would be Right. |
| Alſo | 3 | ∠ABE is Right. *Conſt.* |
| ∴ c. 3. 2. | 4 | ∠ABE would be = ∠ABG. *Ax.* 21. |
| But | 5 | ∠ABE ⊏ ∠ABG. *Ax.* 17. |
| ∴ | 6 | No other Right-line, beſides BE, is the common Section of the two Planes. |
| ∴ | 7 | BE is the common Section of the two Planes. |
| ∴ | 8 | BC, BD, BE, are in the ſame Plane. Q. E. D. |

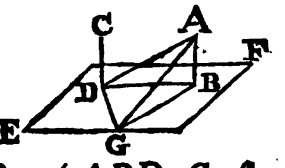
## PROP. VI.

*If two Lines* AB, CD, *be perpendicular to the ſame Plane* EF, *ſo that* ABD, CDB, *be Right-angles, thoſe two Lines are parallel.*

## PREPARATION.

In the Plane EF, draw (11. 1.) GD ⊥ BD and = AB, ſo (*Def.* 10. 1.) is ∠GDB Right, and conſequently (*Ax.* 11.) = ∠ABD, draw alſo BG, AG, AD.

I ſay, AB ∥ CD.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | GD = AB, BD is common, ∠GDB = ∠ABD. *Conſt.* |
| ∵ e. 1. | 2 | AD = BG. 4. 1. |
| Again | 3 | AG is common; GD ⊥ BD; ABD, CDB, are r∠'s. *Conſt.* |
| ∵ e. 1. 2. 3. | 4 | GD = AB, AD = BG, AG is common. |
| ∵ e. 4. | 5 | ∠GDA = ∠ABG. 8. 1. |
| But | 6 | ABG is a r∠. *Def.* 3. |
| ∵ e. 5. 6. | 7 | GDA is a r∠. |
| ∵ e. 7. | 8 | GD ⊥ AD. *Def.* 10. 1. |
| Laſtly | 9 | CDG is a r∠. *Def.* 3. |
| ∵ e. 9. | 10 | GD ⊥ CD. *Def.* 10. 1. |
| ∵ e. 10. 8. 3. | 11 | GD ⊥ CD, AD, BD. |
| ∵ e. 11. | 12 | CD, AD, BD, are in the ſame Plane. 5. |
| Alſo | 13 | AB, AD, BD, are in the ſame Plane. 2. |
| ∵ e. 13. 12. | 14 | AB, CD, are in the ſame Plane. |
| ∵ e. 14. 3. | 15 | AB, CD, are in the ſame Plane, ABD, CDB, are r∠s. |
| ∵ e. 15. | 16 | AB ∥ CD. 28. 1. Q. E. D. |

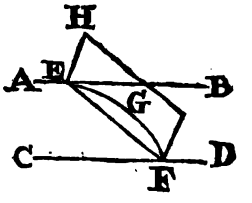
PROP.

## PROP. VII.

*If there be two parallel Right-lines* AB, CD, *and any Points* E, F, *be taken in both of them, the Right-line* EF, *which joins the Points, is in the ſame Plane with thoſe Parallels.*

### PREPARATION.

The Parallels AB, CD, being in one Plane, let the Right-line EF, be in another Plane HF, which cuts the Plane that AB, CD, are in, in the Points E, F, and draw EGF.




I ſay, **EF is in the ſame Plane with AB, CD.**

### DEMONSTRATION.

For 1 If any other Line, beſides EF, ſhould be the common Section of the two Planes; as EGF.

Then 2 That Line EGF, would be a Right-line. 3.

∵ 3 Two Right-lines EF, EGF, would include Space.

But 4 Two Right-lines cannot include Space. *Ax.* 20.

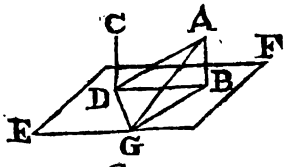
∴ 5 No other Line, beſides EF, is the common Section of the two Planes.

∴ 6 EF is the common Section of the two Planes.

∵ 7 **EF is in the ſame Plane with AB, CD.** Q.E.D.

## PROP. VIII.

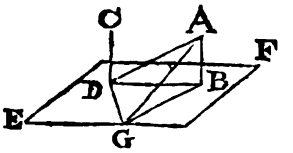
*If of two parallel Lines* AB, CD, *the one* AB *be perpendicular to a Plane* EF, *the other* CD, *will be alſo perpendicular to the ſame Plane.*



## PREPARATION.

Draw the Line BD, cutting the Parallels in the Points B, D, ſo (ABD being (*Def.* 10. 1.) a Right-angle) will the Angle CDB, be (29. 1.) alſo Right. In the Plane EF, draw (11. 1.) GD ⊥ BD and = AB, ſo (*Def.* 10. 1.) is ∠GDB Right, and conſequently (*Ax.* 21.) = ∠ABD, draw alſo BG, AG, AD.

I ſay, ℭ𝔇 ⊥ ℭ𝔉.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | GD = AB, BD is common, ∠GDB = ∠ABD. *Conſt.* |
| ∵ e. 1. | 2 | AD = BG. 4. 1. |
| Again | 3 | AG is common; GDB, CDB, are r∠ˢ. *Conſt.* |
| ∵ e. 1. 2. 3. | 4 | GD = AB, AD = BG, AG is common. |
| ∵ e. 4. | 5 | ∠GDA = ∠ABG. 8. 1. |
| But | 6 | ABG is a r∠. *Def.* 3. |
| ∵ e. 5. 6. | 7 | GDA is a r∠. |
| ∵ e. 3. 7. | 8 | GDB, GDA, are r∠ˢ. |
| ∵ e. 8. | 9 | GD ⊥ BD, AD. *Def.* 10. 1. |
| ∵ e. 9. | 10 | GD ⊥ the Plane of BD, AD. 4. |
| But | 11 | AB, CD, are in the ſame Plane with BD, AD. 7. |
| ∵ e. 10. 11. | 12 | GD ⊥ the Plane of AB, CD. |
| ∵ e. 12. | 13 | CDG is a r∠. *Def.* 3. |
| ∵ e. 3. 13. | 14 | CDB, CDG, are r∠ˢ. |
| ∵ e. 14. | 15 | CD ⊥ BD, GD. *Def.* 10. 1. |
| ∵ e. 15. | 16 | ℭ𝔇 ⊥ ℭ𝔉. 4. Q. E. D. |

## PROP. IX.

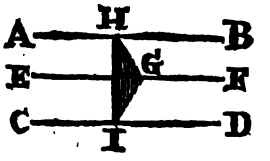
*Right-lines* AB, CD, *which are parallel to the ſame Right-line* EF, *but not in the ſame Plane with it, are parallel one to the other.*

## PREPARATION.

In the Plane of the Parallels AB, EF, draw GH ⊥ EF; alſo in the Plane of the Parallels CD, EF, draw GI ⊥ EF.

I ſay, 𝔄𝔅 ∥ ℭ𝔇.

DEMON-




DEMONSTRATION.

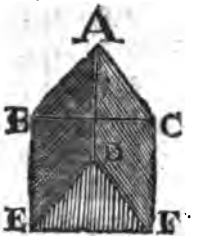
| | | |
|---|---|---|
| | 1 | EG ⊥ GH, GI. *Conſt.* |
| ∵ e. 1. | 2 | EG ⊥ the Plane of GH, GI. 4. |
| Again | 3 | AB ‖ EG. *Hyp.* |
| ∵ e. 3. 2. | 4 | AB ‖ EG, EG ⊥ the Plane of GH, GI. |
| ∵ e. 4. | 5 | AB ⊥ the Plane of GH, GI. 8. |
| Laſtly | 6 | CD ‖ EG. *Hyp.* |
| ∵ e. 6. 2. | 7 | CD ‖ EG, EG ⊥ the Plane of GH, GI. |
| ∵ e. 7. | 8 | CD ⊥ the Plane of GH, GI. 8. |
| ∵ e. 5. 8. | 9 | AB, CD, are ⊥ the Plane of GH, GI. |
| ∵ e. 9. | 10 | AB ‖ CD. 6. Q. E. D. |

## PROP. X.

*If two Right-lines* AB, AC, *touching one another, be parellel to two other Right-lines* DE, DF, *touching one another, but not in the ſame Plane with them, thoſe Right-lines contain equal Angles* BAC, EDF.

PREPARATION.

Make AB = DE, AC = DF, and join BC, EF, BE, AD, CF.




I ſay, ∠BAC = ∠EDF.

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AB ‖ DE, AC ‖ DF. *Hyp.* |
| Alſo | 2 | AB = DE, AC = DF. *Conſt.* |
| ∵ e. 1. 2. | 3 | AB ‖ and = DE, AC ‖ and = DF. |
| ∵ e. 3. | 4 | BE ‖ and = AD, CF ‖ and = AD. 33. 1. |
| ∵ e. 4. | 5 | BE = CF. *Ax.* 1. |
| Alſo e. 4. | 6 | BE ‖ CF. 9. |
| ∵ e. 6. 5. | 7 | BE ‖ and = CF. |
| ∵ e. 7. | 8 | BC = EF. 33. 1. |
| ∵ e. 2. 8. | 9 | AB = DE, AC = DF, BC = EF. |
| ∵ e. 9. | 10 | ∠BAC = ∠EDF. 8. 1. Q. E. D. |

## PROP. XI.

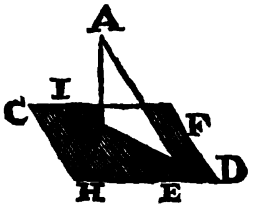
*From a Point given* A, *without a Plane* CD, *to let fall a Line* AB, *perpendicular to the ſaid Plane.*

## PREPARATION.

In the Plane CD, draw any Line EF; upon which, from the Point A, let fall (12. 1.) the ⊥ AG; and from the Point G, in the Plane CD, draw (11. 1.) the ⊥ GI, upon which, let fall (12. 1.) the ⊥ AB, and thro' the Point B, draw (31. 1.) HB ‖ EG.

I ſay, AB ⊥ CD.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | EG ⊥ AG, GB. *Conſt.* |
| ∴ e. 1. | 2 | EG ⊥ the Plane AGB. 4. |
| Again | 3 | HB ‖ EG. *Conſt.* |
| ∴ e. 3. 2. | 4 | HB ‖ EG, EG ⊥ the Plane AGB. |
| ∴ e. 4. | 5 | HB ⊥ the Plane AGB. 8. |
| ∴ e. 5. | 6 | ABH is a r∠. *Def.* 3. |
| ∴ e. 6. | 7 | AB ⊥ HB. *Def.* 10. 1. |
| Alſo | 8 | AB ⊥ GB. *Conſt.* |
| ∴ e. 7. 8. | 9 | AB ⊥ HB, GB. |
| ∴ e. 9. | 10 | AB ⊥ CD. 4. Q. E. D. |

PROP.

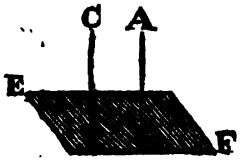
## PROP. XII.

*From a Point given* B, *in a Plane* EF, *to erect a Right-Line* AB, *perpendicular to the ſaid Plane.*

### PREPARATION.

Upon the Plane EF, let fall (11.) any perpendicular CD, and thro' the Point B, draw (31. 1.) AB || CD.

I ſay, AB ⊥ EF.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AB || CD, CD ⊥ EF. *Conſt.* |
| ∴ c. 1. | 2 | AB ⊥ EF. 8. Q. E. D. |

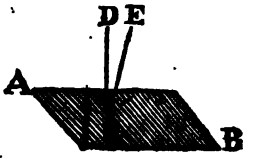
## PROP. XIII.

*Two Right-lines, cannot be drawn perpendicular to a Plane, thro' the ſame Point.*

### PREPARATION.

Thro' the Point C, in the Plane AB, draw the Right-lines CD, CE.

I ſay, CD and CE are not ⊥ AB.




### DEMONSTRATION.

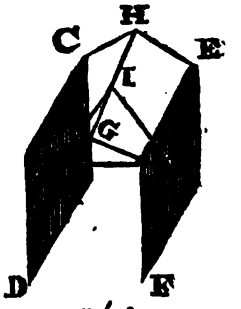
| | | |
|---|---|---|
| For | 1 | If CD and CE were ⊥ AB. |
| Then | 2 | CD would be || CE. 6. |
| But | 3 | CD is not || CE. *Def.* 34. 1. |
| ∴ | 4 | CD and CE are not ⊥ AB. Q. E. D. |

## PROP. XIV.

*Planes* CD, EF, *to which the ſame Right-line* AB *is Perpendicular, are parallel.*

## PREPARATION.

If CD, EF, be ſaid not to be parallel, ſuppoſe them to be produced till they meet in ſome Line GH, in which taking any Point I, draw to it the Right-lines AI, BI, in the ſaid Planes; ſo (*Def.* 3.) will IAB, IBA, be Right-angles.




I ſay, CD ∥ EF.

## DEMONSTRATION.

For 1 If CD, EF, being produced, ſhould ever meet; as in GH.

Then 2 IAB, IBA, two ∠ˢ of a △, would be = 2 r∠ˢ.

But 3 Two ∠ˢ of a △ are ⌐ 2 r∠ˢ. 17. 1.

∴ 4 CD, EF, being produced, will never meet; as in GH.

∴ 5 CD ∥ EF. *Def.* 7. Q. E. D.

## PROP. XV.

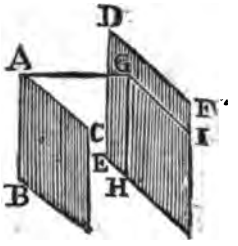
*If two Lines* AB, AC, *touching one the other, be parallel to two others* DE, DF, *touching one the other, but not in the ſame Plane with them, their Planes* BC, EF, *are alſo parallel.*

## PREPARATION.

From the Point A, draw (11.) AG ⊥ EF, and thro' the Point G, where it meets the Plane, draw (31. 1.) in the ſame Plane, the Right-lines GH, GI, parallel to DE, DF, ſo (9.) are they alſo parallel to AB, AC; moreover HGA, IGA, are (*Def.* 3.) Right-angles.

I ſay,

I ſay, BC || EF.



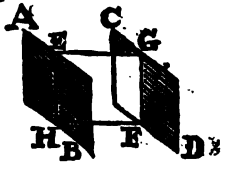

DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | GH \|\| AB, HGA is a r∠. *Conſt.* |
| ∵ e. 1. | 2 | BAG is a r∠. 29. 1. |
| Alſo | 3 | GI \|\| AC, IGA is a r∠. *Conſt.* |
| ∵ e. 3. | 4 | CAG is a r∠. 29. 1. |
| ∵ e. 2. 4. | 5 | BAG, CAG, are r∠s. |
| ∵ e. 5. | 6 | AG ⊥ AB, AC. *Def.* 10. 1. |
| ∵ e. 6. | 7 | AG ⊥ BC. 4. |
| Alſo | 8 | AG ⊥ EF. *Conſt.* |
| ∵ e. 7. 8. | 9 | AG ⊥ BC, EE. |
| ∵ e. 9. | 10 | BC \|\| EF. 14. Q. E. D. |

## PROP. XVI.

*If two parallel Planes* AB, CD, *be cut by ſome other Plane* EF, *their common Sections are alſo parallel.*

That is, EH || GF.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | EH, GF, being produced, will ſtill be in the Planes AB, CD, produced. 1. |
| But | 2 | The Planes AB, CD, being produced, will never meet. *Def.* 7. |
| ∵ | 3 | EH, GF, being produced, will never meet. |
| ∵ | 4 | EH \|\| GF. *Def.* 34. 1. Q. E. D. |

PROP.

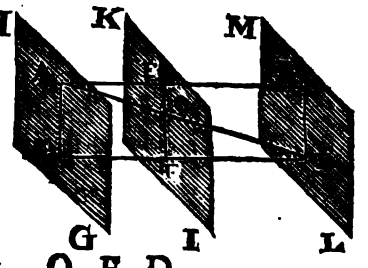
## PROP. XVII.

*If two Right-lines* AB, CD, *be cut by parallel Planes* GH, IK, LM; *they ſhall be cut proportionally, ſo that* AE:EB::CF:FD.

### PREPARATION.

In the Planes GH, IK, LM, draw the Right-lines AC, EF, BD, as alſo AD, paſſing thro' the Plane IK, in the Point O.

I ſay, AE:EB::CF:FD.




### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | EO ‖ BD. 16. |
| ∴ e. 1. | 2 | AE:EB::AO:OD. 2. 6. |
| Alſo | 3 | OF ‖ AC. 16. |
| ∴ e. 3. | 4 | CF:FD::AO:OD. 2. 6. |
| ∴ e. 2. 4. | 5 | AE:EB::CF:FD. 11. 5. Q. E. D. |

## PROP. XVIII.

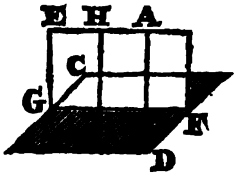
*If a Right-line* AB, *be perpendicular to a Plane* CD, *all the Planes in which that Line is found, are perpendicular to the ſame Plane.*

### PREPARATION.

Thro' AB, draw any Plane EF, making the Section GF, with the Plane CD; from ſome Point whereof I, draw (11. 1.) in the Plane EF, the ⊥ HI, ſo (ABI, HIB, being (*Def.* 10. 1.) Right-angles) is HI (28. 1.) ‖ AB.

I ſay

I say, **All the Planes, in which AB is found, are ⊥ CD.**




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | HI ‖ AB, AB ⊥ CD. *Const.* |
| ∴ c. 1. | 2 | HI ⊥ CD. 8. |
| ∴ c. 2. | 3 | EF ⊥ CD. *Def.* 4. |
| Also | 4 | After the same manner, are all the other Planes, in which AB is found, ⊥ CD. |
| ∴ | 5 | **All the Planes, in which AB is found, are ⊥ CD.** Q. E. D. |

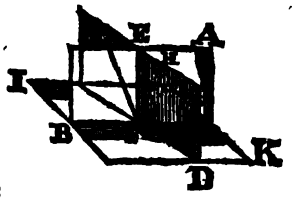
## PROP. XIX.

*If two Planes* AB, ED, *which cut each other, be perpendicular to another Plane* IK, *their common Section* EF, *will be perpendicular to the same Plane.*

## PREPARATION.

From the Point F, draw (11. 1.) in the Planes AB, ED, the Right-lines FG, FH, perpendicular to the common Sections FB, FD, so (AB and ED being *(Hyp.)* ⊥ IK) are FG and FH also *(Def. 4.)* ⊥ IK.

I say, **The common Section EF ⊥ IK.**




DEMONSTRATION.

| | | |
|---|---|---|
| For | 1 | Either ⊥ FG, FH, will coincide in the common Section, and make one ⊥ EF. |
| Or | 2 | Two Right-lines can be drawn ⊥ a Plane, thro' the same Point. |
| But | 3 | Two Right-lines cannot be drawn ⊥ a Plane, thro' the same Point. 13. |
| ∴ | 4 | ⊥ FG, FH, will coincide in the common Section, and make one ⊥ EF. |
| ∴ | 5 | **The common Section EF ⊥ IK.** Q. E. D. |

## PROP. XX.

*If a Solid Angle* A, *be contained under three Plane Angles* BAC, BAD, CAD, *any two of them howſoever taken, are greater than that which remains; ſo that the Sum of the Leſſer Angles* CAD, BAD, *is greater than the greateſt* BAC.

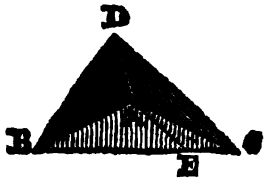
## PREPARATION.

From the greateſt BAC, take (23. 1.) ∠BAE = ∠BAD, and making AD = AE, draw BEC, BD, DC.

I ſay, ∠CAD + ∠BAD ⊏ ∠BAC.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AD = AE, AB is common, ∠BAE = ∠BAD. *Conſt.* |
| ∵ c. 1. | 2 | BE = BD. 4. 1. |
| Again | 3 | BD + DC ⊏ BC. 20. 1. |
| ∵ c. 2. 3. | 4 | BE + DC ⊏ BC. *Sub.* |
| i. e. | 5 | BE + DC ⊏ BE + EC. |
| 5 — BE. | 6 | DC ⊏ EC. *Ax.* 16. |
| Laſtly | 7 | AC is common |
| ∵ c. 1. 7. 6. | 8 | AD = AE, AC is common, DC ⊏ EC. |
| ∵ c. 8. | 9 | ∠CAD ⊏ ∠CAE. 25. 1. |
| 9 + ∠BAD. | 10 | ∠CAD + ∠BAD ⊏ ∠CAE + ∠BAD. *Ax.* 15. |
| ∵ c. 1. 10. | 11 | ∠CAD + ∠BAD ⊏ ∠CAE + ∠BAE. *Sub.* |
| i. e. | 12 | ∠CAD + ∠BAD ⊏ ∠BAC. Q. E. D. |




PROP.

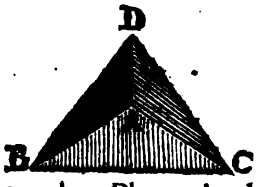
## PROP. XXI.

*Every Solid Angle* A, *is contained under leſs Angles than four Right.*

### PREPARATION.

Having drawn a Plane, cutting the Sides of the Solid Angle any-wiſe, and making as many Triangles A B C, A C D, A D B, as the Plane hath Sides; denote all the Angles of the Plane by X, and the Sum of all the Angles at the Baſes of the Triangles (every two of which, together with a Correſpondent Angle of the Plane, form a Solid Angle) by Z.

I ſay, $\mathfrak{A} \sqsupset 4 \mathfrak{r}\angle^{s}$.



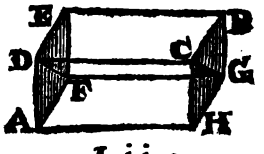

### DEMONSTRATION.

1. X = twice as many $\mathfrak{r}\angle^{s}$, bating four, as the Plane hath Sides. *The.* 1. 1.
2. ∵ c. 1. X + 4 $\mathfrak{r}\angle^{s}$ = twice as many $\mathfrak{r}\angle^{s}$, as the Plane hath Sides.
3. Alſo Z + A = twice as many $\mathfrak{r}\angle^{s}$, as the Plane hath Sides. 32. 1.
4. ∵ c. 3. 2. Z + A = X + 4 $\mathfrak{r}\angle^{s}$. *Ax.* 1.
5. But Z ⊏ X. 20.
6. ∵ c. 4. 5. $\mathfrak{A} \sqsupset 4 \mathfrak{r}\angle^{s}$. Q. E. D.

*The* 22*d and* 23*d Propoſitions are needleſs.*

## PROP. XXIV.

*If a Solid* A B, *be contained under parallel Planes, the oppoſite Planes thereof, are ſimilar and equal Parallelograms.*



## PREPARATION.

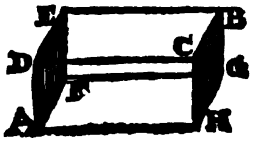
Draw DF, CG.

I ſay, **The oppoſite Planes are ſimilar and equal ▭'.**

## DEMONSTRATION.

1 Each Plane cuts four others, whereof the Oppoſites are parallel.

∴ e. 1. 2 The common Sections of each Plane with the oppoſite Planes are parallel. 16.




∴ e. 2. 3 Each Plane is a ▭. *Def.* 35. 1.

∴ e. 3. 4 AF = HG, AD = HC. 34. 1.

Alſo e. 3. 5 AF ‖ HG, AD ‖ HC. *Def.* 35. 1.

∴ e. 5. 6 ∠DAF = ∠CHG. 10.

∴ e. 4. 6. 7 AF = HG, AD = HC, ∠DAF = ∠CHG.

∴ e. 7. 8 △' DAF, CHG, are equiangular and equal. 4. 1.

∴ e. 8. 9 △' DAF, CHG, are ſimilar and equal. *Scho.* 1. 4. 6.

∴ e. 9. 10 ▭' AE, HB, their Doubles, are ſimilar and equal.

Alſo 11 After the ſame manner, are the other oppoſite Planes, ſimilar and equal ▭'.

∴ 12 **The oppoſite Planes are ſimiler and equal ▭'** Q. E. D.

## PROP. XXV.

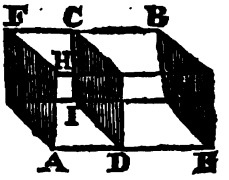
*If a Parallelepipedon* AB, *be cut by a Plane* CD, *parallel to the oppoſite Planes* AF, EB; *the Solids* AC, EC, *which are form'd by that Diviſion, are to one another as their Baſes* AI, EI.

PRE-

## PREPARATION.

Imagine Planes parallel to the common Baſe AG, to paſs thro' all the Points of the Line AH, that may be taken for the common Height of the two Solids AC, EC; and theſe Planes will divide the Solids into little Planes or Surfaces, equal in number in both, of which (*Def.* 1.) they are ſuppoſed to conſiſt.

I ſay, 𝔄𝔍 : 𝔈𝔍 :: 𝔄𝔊 : 𝔈𝔊.




## DEMONSTRATION.

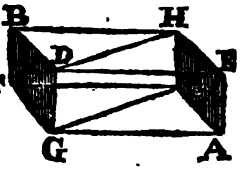
| | | |
|---|---|---|
| | 1 | Each Plane of each Solid is = its Baſe. 24. |
| ∴ e. 1. | 2 | AI : EI :: each Plane of AC : each Plane of EC. |
| ∴ e. 2. | 3 | AI : EI :: all the Planes of AC : all the Planes of EC. 12. 5. |
| i. e. | 4 | 𝔄𝔍 : 𝔈𝔍 :: 𝔄𝔊 : 𝔈𝔊. Q. E. D. |

*The 26th and 27th Propoſitions are needleſs.*

## PROP. XXVIII.

*A Parallelepipedon* AB, *is divided into two equal Priſms* BDHFGC, EDHAGC, *by a Plane* DC, *which paſſes thro' the two Diagonals* GC, DH, *of the oppoſite Planes* FA, BE.

That is, 𝔅𝔇𝔎𝔉𝔊𝔊 = 𝔈𝔇𝔎𝔄𝔊𝔊.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | DC is common. *Conſt.* |
| And | 2 | BC and DA, BG and HA, are equal. 24. |
| Alſo | 3 | BE and FA, are equal. 24. |
| ∴ e. 3. | 4 | BDH and EDH, FGC and AGC, their halves, are equal. |
| ∴ e. 1. 2. 4. | 5 | DC is common, BC and DA, BG and HA, BDH and EDH, FGC and AGC, are equal. |
| ∴ e. 5. | 6 | 𝔅𝔇𝔎𝔉𝔊𝔊 = 𝔈𝔇𝔎𝔄𝔊𝔊. *Def.* 12. Q. E. D. |

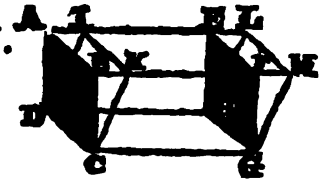
## PROP. XXIX, and XXX.

*Solid Parallelepipedons* CBADHEFG, CKIDHLMG, *being conſtituted upon the ſame Baſe* CDHG, *and in the ſame Height, that is, between the ſame parallel Planes* AM, DG, *are equal one to the other.*

*Caſe* 1 If the Parallelepipedons CBADHEFG, CKIDHLMG, have their inſiſting Lines CB, CK, GF, GM, DA, DI, HE, HL, placed within the ſame parallel Planes CBMG, DALH.

I ſay, ▱CBADHEFG = ▱CKIDHLMG.

DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | DE, DL, are ▱ˢ. 24. |
| ∵ e. 1. | 2 | AD = EH, ID = LH. 34. 1. |
| Alſo e. 1. | 3 | AE = DH = IL. 34. 1. |
| ∵ e. 3. | 4 | AE = IL. *Ax.* 1. |
| ∴ — IE. | 5 | AE — IE = IL — IE. *Ax.* 12. |
| i. e. | 6 | AI = EL. |
| ∵ e. 2. 6. | 7 | AD = EH, ID = LH, AI = EL. |
| ∵ e. 7. | 8 | △ADI = △EHL. 8. 1. |
| Again | 9 | After the ſame Manner is △BCK = △FGM. |
| Laſtly | 10 | ▱AK = ▱EM. 36. 1. |
| Alſo | 11 | ▱DB = ▱HF, ▱DK = ▱HM. 24. |
| ∵ e. 8, 9, 10, 11. | 12 | △ADI = △EHL, △BCK = △FGM, ▱AK = ▱EM, ▱DB = ▱HF, ▱DK = ▱HM. |
| ∵ e. 12. | 13 | Priſm ABCDIK = Priſm EFGHLM. *Def.* 12. |
| 13 + IDCKFEHG | 14 | ABCDIK + IDCKFEHG = EFGHLM + IDCKFEHG. *Ax.* 11. |
| i. e. | 15 | ▱CBADHEFG = ▱CKIDHLMG. Q. E. D. |

If

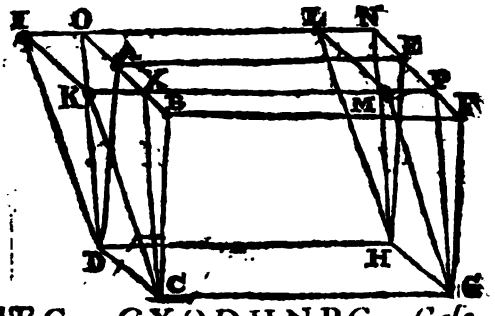
*Case* 2 If the Parallelepipedons CBADHEFG, CKIDHLMG, have not their inſiſting Lines CB, CK, GF, GM, DA, DI, HE, HL, placed within the ſame parallel Planes.

PREPARATION.

Having produced IL, FE, which being (*Hyp.*) in the ſame Plane will meet in the Point N, as alſo BA, KM, which being *(Hyp.)* in the ſame Plane will cut each other in the Point X, and meet IL, FE, (which *(Hyp.)* are in the ſame Plane with them) in the Points O, P, join CX, DO, HN, GP; ſo will the oppoſite Quadrilateral Figures PGHN, XCDO, be parallel, they being in the ſame Planes with FGHE, BCDA, which are *(Def.* 10.) Parallel; alſo the oppoſite Quadrilateral Figures ODHN, XCGP, will be parallel, they being in the ſame Planes with IDHL, KCGM, which are *(Def.* 10.) Parallel; moreover the Quadrilateral Figure XONP, will be parallel to its oppoſite Quadrilateral Figure CDHG, it being in the ſame Plane with BAEF, which is (*Def.* 10.) parallel to CDHG, and conſequently the Solid CXODHNPG, which has its oppoſite Sides parallel, will be (*Def.* 10.) a Parallelepipedon.

I ſay, CBADHEFG = CKIDHLMG.

DEMONSTRATION.




1 CBADHEFG = CXODHNPG. *Case* 1.
Alſo 2 CKIDHLMG = CXODHNPG. *Case* 1.
∴ 1. 2. 3 CBADHEFG = CKIDHLMG. *Ax.* 1. Q.E.D.

### COROLLARY.

Triangular Priſms, which (28.) are half Parallelepipedons, being conſtituted on the ſame Baſe, and in the ſame Height, are (*Ax.* 8.) equal.

'Tis the ſame alſo in Polygonal Priſms, becauſe they may be reduced into Triangular Priſms.

### PROP. XXXI.

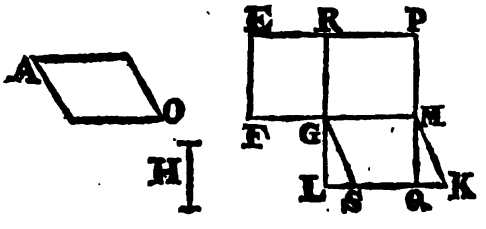
*Solid Parallelepipedons* EGH, AOH, *being conſtituted upon equal Baſes* EG, AO, *and in the ſame Height* H, *are equal one to the other.*

*Caſe* 1 If the Parallelepipedons EGH, AOH, have the Sides perpendicular to the Baſes.

### PREPARATION.

To the Side FG produced, apply a Parallelogram GK, equal and ſimilar to the Parallelogram AO, and having compleated the Parallelogram GP, let the Right-lines PM, RG, meet KS, in Q and L, and imagine Parallelepipedons whoſe Sides are Perpendicular to their Baſes, and whoſe common Height is H, to be conſtituted on GK, GQ, GP.

I ſay, ☾GH = ☽AH.




DEMON-

## DEMONSTRATION.

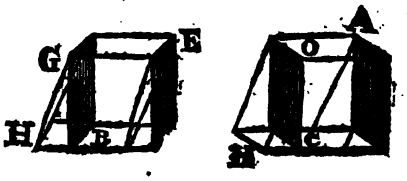
| | | |
|---|---|---|
| | 1 | EGH:GPH::EG:GP. 25. |
| And | 2 | AO=EG. *Hyp.* |
| ∵ e. 1. 2. | 3 | EGH:GPH::AO:GP. *Sub.* |
| Again | 4 | GK=AO. *Conſt.* |
| ∵ e. 3. 4. | 5 | EGH:GPH::GK:GP. *Sub.* |
| And | 6 | GQ=GK. 35. 1. |
| ∵ e. 5. 6. | 7 | EGH:GPH::GQ:GP. *Sub.* |
| Alſo | 8 | GQH:GPH::GQ:GP. 25. |
| ∵ e. 7. 8. | 9 | EGH:GPH::GQH:GPH. 11. 5. |
| ∵ e. 9. | 10 | EGH=GQH. 9. 5. |
| Alſo | 11 | GKH=GQH. *Caſe* 1. 29 and 30. |
| ∵ e. 10. 11. | 12 | EGH=GKH. *Ax.* 1. |
| Laſtly | 13 | AO is equal and ſimilar to GK. *Conſt.* |
| ∵ e. 13. | 14 | AOH=GKH. *Def.* 12. |
| ∵ e. 12. 14. | 15 | ℭ𝔊𝔅=𝔄𝔒𝔅. *Ax.* 1. Q. E. D. |

*Caſe.* 2 If the Parallelepipedons EGH, AOH, have not the Sides perpendicular to the Baſes.

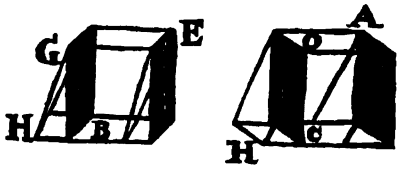
## PREPARATION.

On the equal Baſes EG, AO, and in the ſame Height, place the Parallelepipedons EGB, AOC, whoſe Sides are perpendicular to the Baſes.

I ſay, ℭ𝔊𝔅=𝔄𝔒𝔅.



## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | EGB = EGH. *Caſe* 1. 29 and 30. |
| Alſo | 2 | EGB = AOC. *Caſe* 1. |
| ÷ e. 1. 2. | 3 | EGH = AOC. *Ax.* 1. |
| Alſo | 4 | AOH = AOC. *Caſe* 1. 29 and 30. |
| ∴ e. 3. 4. | 5 | ℭ𝔊ℌ = 𝔄𝔒ℌ. *Ax.* 1. Q. E. D. |

## COROLLARY.

Triangular Priſms, which (28.) are half Parallelepipedons, being conſtituted on equal Baſes, and in the ſame Height, are (*Ax.* 8.) equal.

'Tis the ſame alſo in Polygonal Priſms, becauſe they may be reduced into Triangular Priſms.

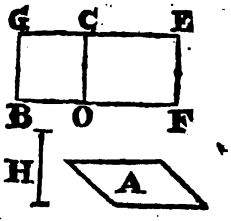
## PROP. XXXII.

*All Parallelepipedons of the ſame Height, are one to the other, as their Baſes.*

## PREPARATION.

Let their Baſes be A and BC, and their common Height H; ſo may the Parallelepipedons be expreſſed by AH and BCH. Upon CO make (44. 1.) the Parallelogram OE equal to the Parallelogram A, having the Angle OCE equal to the given Angle G. Upon OE, BC, imagine Parallelepipedons OEH, BCH, to be placed, of the ſame Height H; ſo will they be Parts of the one Parallelepipedon BEH. I ſay,

I ſay, 𝔄𝔊 : 𝔅ℭ𝔊 :: 𝔄 : 𝔅ℭ.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | OEH : BCH :: OE : BC. 25. |
| And | 2 | OE = A. *Conſt.* |
| ∴ e. 1. 2. | 3 | OEH : BCH :: A : BC. *Sub.* |
| Again e. 2. | 4 | OEH = AH. 31. |
| ∴ e. 3. 4. | 5 | 𝔄𝔊 : 𝔅ℭ𝔊 :: 𝔄 : 𝔅ℭ. *Sub.* Q. E. D. |

## COROLLARY.

Triangular Priſms, which (28.) are half Parallelepipedons, of the ſame Height, are to one another as their Baſes.

'Tis the ſame alſo in Polygonal Priſms, becauſe they may be reduced into Triangular Priſms.

## LEMMA I.

*If four Quantities* A, AR, ARR, ARRR, *be continually proportional, that is, if* A : AR :: AR : AR² :: AR² : AR³; *then as the Firſt is to the Fourth, ſo is the Cube of the Firſt, to the Cube of the Second.*

That is, 𝔄 : 𝔄𝔯³ :: 𝔄³ : 𝔄³𝔯³.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | A : AR :: ARR : ARRR. *Def.* 2. 5. |
| And | 2 | A × A³R³ = A³ × AR³. |
| ∴ e. 2. | 3 | 𝔄 : 𝔄𝔯³ :: 𝔄³ : 𝔄³𝔯³. *Lem.* 2. 5. Q. E. D. |

## LEMMA II.

*If four Quantities* A, AR, ARR, ARRR, *or* AR, A, BR, B, *either continued or diſcontinued, be proportional, their Cubes are alſo proportional.*

That is, $\mathfrak{A}^3 : \mathfrak{A}^3\mathfrak{R}^3 :: \mathfrak{A}^3\mathfrak{R}^6 : \mathfrak{A}^3\mathfrak{R}^9$, and $\mathfrak{A}^3\mathfrak{R}^3 : \mathfrak{A}^3 :: \mathfrak{B}^3\mathfrak{R}^3 : \mathfrak{B}^3$.

## DEMONSTRATION.

1 A : AR :: ARR : ARRR. and AR : A :: BR : B. *Def.* 2. 5.
And 2 $A^3 \times A^3R^9 = A^3R^3 \times A^3R^6$. and $A^3R^3 \times B^3 = A^3 \times B^3R^3$.
∴ e. 2. 3 $\mathfrak{A}^3 : \mathfrak{A}^3\mathfrak{R}^3 :: \mathfrak{A}^3\mathfrak{R}^6 : \mathfrak{A}^3\mathfrak{R}^9$, and $\mathfrak{A}^3\mathfrak{R}^3 : \mathfrak{A}^3 :: \mathfrak{B}^3\mathfrak{R}^3 : \mathfrak{R}^3$. *Lem.* 2. 5. Q. E. D.

## SCHOLIUM.

Becauſe (*Lem.* 2. 5.) $A^2 : A^2R^2 :: A^2R^4 : A^2R^6$. and $A^2R^2 : A^2 :: B^2R^2 : B^2$; therefore if four Quantities, either continued or diſcontinued, be proportional, their Squares are alſo proportional.

## PROP. XXXIII.

*Similar Solid Parallelopipedons* EB, EN, *are in the Triplicate Ratio, or as the Cubes, of their Homologous Sides* EA, EF.

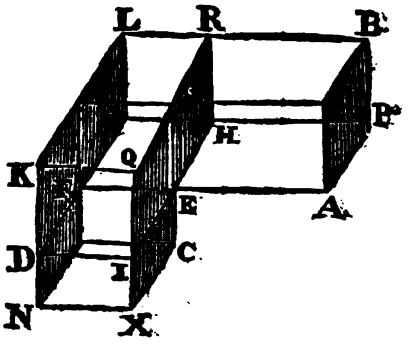
## PREPARATION.

The Parallelepipedons EB, EN, being *(Hyp.)* ſimilar, all their Planes will (*Def.* 12.) be ſimilar, and conſequently (*Def.* 1. 6.) EA : EF :: EH : EI, and EH : EI :: EG : EC. Alſo the Angles of their Planes will (*Def.* 1. 6.) be equal. Let therefore the Parallelepipedons EB, EN, be ſo placed, that the equal Angles HEA, FEI, as alſo the equal Angles

gles GEH, IEC, may be vertically Oppoſite in the ſame Planes; ſo (*Scho.* 15. 1.) will FEA, IEH, GEC, be ſtraight Lines. Imagine upon the Planes EO, ED, Solids EL, EK, ſo placed, as that the Solids EB, EL, may make one Parallelepipedon AL; the Solids EL, EK, another Parallelepipedon IL; and the Solids EK, EN, a third Parallelepipedon GN.

I ſay, $\mathfrak{E}'\mathfrak{A} : \mathfrak{E}'\mathfrak{F} :: \mathfrak{E}\mathfrak{B} : \mathfrak{E}\mathfrak{N}$.

## DEMONSTRATION.




| | | | |
|---|---|---|---|
| | 1 | EA : EF :: EP : EO. | 1. 6. |
| Alſo | 2 | EB : EL :: EP : EO. | 25. |
| ∵ e. 1. 2. | 3 | EA : EF :: EB : EL. | 11. 5. |
| Alſo | 4 | EA : EF :: EH : EI. | *Def.* 1. 6. |
| ∵ e. 3. 4. | 5 | EB : EL :: EH : EI. | 11. 5. |
| Alſo | 6 | EO : ED :: EH : EI. | 1. 6. |
| ∵ e. 5. 6. | 7 | EB : EL :: EO : ED. | 11. 5. |
| Alſo | 8 | EL : EK :: EO : ED. | 25. |
| ∵ e. 7. 8. | 9 | EB : EL :: EL : EK. | 11. 5. |
| Again | 10 | EH : EI :: ER : EQ. | 1. 6. |
| Alſo | 11 | EL : EK :: ER : EQ. | 25. |
| ∵ e. 10. 11. | 12 | EH : EI :: EL : EK. | 11. 5. |
| Alſo | 13 | EH : EI :: EG : EC. | *Def.* 1. 6. |
| ∵ e. 12. 13. | 14 | EL : EK :: EG : EC. | 11. 5. |
| Alſo | 15 | EQ : EX :: EG : EC. | 1. 6. |
| ∵ e. 14. 15. | 16 | EL : EK :: EQ : EX. | 11. 5. |
| Alſo | 17 | EK : EN :: EQ : EX. | 25. |
| ∵ e. 16. 17. | 18 | EL : EK :: EK : EN. | 11. 5. |
| ∵ e. 9. 18. | 19 | EB : EL :: EL : EK :: EK : EN. | 11. 5. |
| ∵ e. 19. | 20 | EB : EN :: E'B : E'L. | *Lem.* 1. |
| Alſo e. 3. | 21 | E'A : E'F :: E'B : E'L. | *Lem.* 2. |
| ∵ e. 21. 20. | 22 | $\mathfrak{E}'\mathfrak{A} : \mathfrak{E}'\mathfrak{F} :: \mathfrak{E}\mathfrak{B} : \mathfrak{E}\mathfrak{N}$. | 11. 5. Q. E. D. |

## COROLLARIES.

1 Becauſe in continual Proportionals, as the Firſt is to the Fourth, ſo (*Lem.* 1.) is the Cube of the Firſt to the Cube of the Second; therefore if four Lines be continually Proportional, as the Firſt is to the Fourth, ſo is a Parallelepipedon deſcrib'd on the Firſt, to a Similar Parallelepipedon deſcrib'd on the Second.

2 Similar Triangular Priſms, which (28.) are the Halves of Similar Parallelepipedons, are in the Triplicate Ratio, or as the Cubes, of their Homologous Sides.

'Tis the ſame alſo in Similar Polygonal Priſms, becauſe they may be reduced into Triangular Priſms.

## PROP. XXXIV.

*Equal Parallelepipedons* AB, CD, *have their Baſes and Altitudes reciprocal, ſo that* AE:CF::CH:AG. *And thoſe Parallelepipedons* AB, CD, *which have their Baſes and Altitudes reciprocal, are equal.*

*Hyp.* 1 Suppoſe AB = CD.

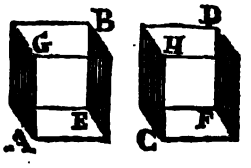
*Caſe.* 1 If the Parallelepipedons AB, CD, have their Sides perpendicular to their Baſes, and their Altitudes CH, AG, equal

## PREPARATION.

CH being (*Hyp.*) = AG, and AB = CD, the Ratio of CH to AG, is the ſame with that of AB to CD, and conſequently (*Def.* 2. 5.) CH:AG::AB:CD.

I ſay,

I say, AE : CF :: CH : AG.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AG = CH. *Hyp.* |
| ∴ e. 1. | 2 | AE : CF :: AB : CD. 32. |
| Also | 3 | CH : AG :: AB : CD. *Def.* 2. 5. |
| ∴ e. 2. 3. | 4 | AE : CF :: CH : AG. 11. 5. Q. E. D. |

*Case* 2 If the Parallelepipedons AB, CD, have their Sides perpendicular to their Bases, and their Altitudes CH, AG, unequal.

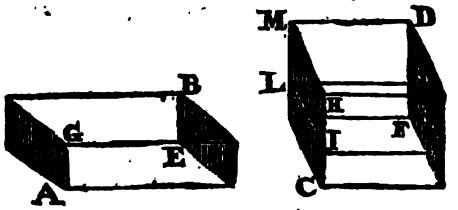
## PREPARATION.

From the Greater CH, take CI = AG; and through I, draw the Plane IK ‖ CF.

I say, AE : CF :: CH : AG.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AE : CF :: AB : CK. 32. |
| And | 2 | AB = CD. *Hyp.* |
| ∴ e. 1. 2. | 3 | AE : CF :: CD : CK. *Sub.* |
| Also | 4 | CM : CL :: CD : CK. 25. |
| ∴ e. 3. 4. | 5 | AE : CF :: CM : CL. 11. 5. |
| Also | 6 | CH : CI :: CM : CL. 1. 6. |
| ∴ e. 5. 6. | 7 | AE : CF :: CH : CI. 11. 5. |
| Lastly | 8 | CI = AG. *Const.* |
| ∴ e. 7. 8. | 9 | AE : CF :: CH : AG. *Sub.* Q. E. D. |



*Case* 3 If the equal Parallelepipedons, have not their Sides perpendicular to their Baſes, their Baſes and Altitudes will be reciprocal.

## DEMONSTRATION.

Imagine Parallelepipedons, which have their Sides perpendicular to their Baſes, to be placed on the Baſes of the Parallelepipedons, which have not their Sides perpendicular to their Baſes, of the ſame Altitude with them; ſo (29 and 30) will they be equal to them. Since therefore the Parallelepipedons, which have their Sides perpendicular to their Baſes, have *(Case* 1 and 2.) their Baſes and Altitudes reciprocal, the Parallelepipedons which have not their Sides perpendicular to their Baſes, but which are their Equals, will have alſo their Baſes and Altitudes reciprocal. Q. E. D.

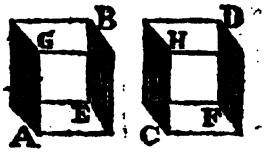
*Hyp.* 2 Suppoſe AE : CF :: CH : AG.

*Case* 1 If the Parallelepipedons AB, CD, have their Sides perpendicular to their Baſes, and their Altitudes, CH, AG, equal.

I ſay, AB = CD.

DEMON-

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | AG = CH. *Hyp.* |
| ∴ e. 1. | 2 | AE : CF :: AB : CD. 32. |
| Alſo | 3 | AE : CF :: CH : AG. *Hyp.* |
| ∴ e. 2. 3. | 4 | AB : CD :: CH : AG. 11. 5. |
| ∴ e. 4. | 5 | AB : CH :: CD : AG. 16. 5. |
| ∴ a. 1. 5. | 6 | AB : AG :: CD : AG. *Sub.* |
| ∴ e. 6. | 7 | AB = CD. 9. 5. Q. E. D. |

*Caſe* 2 If the Parallelepipedons AB, CD, have their Sides perpendicular to their Baſes, and their Altitudes CH, AG, unequal.

## PREPARATION.

From the greater CH, take CI = AG, and through I, draw the Plane IK ∥ CF.

I ſay, AB = CD.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AE : CF :: CH : AG. *Hyp.* |
| And | 2 | CI = AG. *Conſt.* |
| ∴ e. 1. 2. | 3 | AE : CF :: CH : CI. *Sub.* |
| Alſo | 4 | AE : CF :: AB : CK. 32. |
| ∴ e. 3. 4. | 5 | CH : CI :: AB : CK. 11. 5. |
| Alſo | 6 | CH : CI :: CM : CL. 1. 6. |
| ∴ e. 5. 6. | 7 | AB : CK :: CM : CL. 11. 5. |
| Alſo | 8 | CD : CK :: CM : CL. 32. |
| ∴ e. 7. 8. | 9 | AB : CK :: CD : CK. 11. 5. |
| ∴ e. 9. | 10 | AB = CD. 9. 5. Q. E. D. |

*Case* 3 If the Parallelepipedons have not their Sides perpendicular to their Baſes, and have their Baſes and Altitudes reciprocal, they will be equal.

## DEMONSTRATION.

Imagine Parallelepipedons, which have their Sides perpendicular to their Baſes, to be placed on the Baſes of the Parallelepipedons, which have not their Sides perpendicular to their Baſes, of the ſame Altitude with them; ſo (29 and 30.) will they be equal to them. Since therefore the Parallelepipedons, which have their Sides perpendicular to their Baſes, are (*Caſe* 1 and 2.) equal, the Parallelepipedons which have not their Sides perpendicular to their Baſes, but which are their Equals, will be alſo equal. Q. E. D.

## COROLLARY.

Equal Triangular Priſms, which (28.) are half Parallelepipedons, have their Baſes and Altitudes reciprocal. And thoſe Triangular Priſms, which have their Baſes and Altitudes reciprocal, are equal.

'Tis the ſame alſo in Polygonal Priſms, becauſe they may be reduced into Triangular Priſms.

*The 25th Propoſition is needleſs.*

## PROP. XXXVI.

*If three Right-lines* A B, A C, A D, *be continually proportional, that is, if* A B : A C :: A C : A D, *the Parallelepipedon* A E, *made of theſe three Lines, is equal to a Parallelepipedon* F H, *which is Equiangular, and has all its Sides equal to the middle Line.*

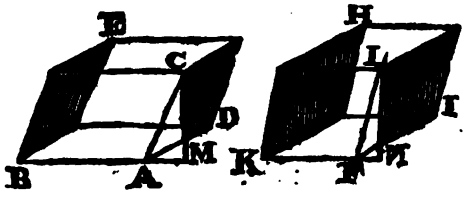
PRE-

## PREPARATION.

From the Points C, L, draw the Lines CM, LN, perpendicular to the Planes of the Baſes; ſo (the Solid Angles A, F, being ſuppoſed equal, ſo that if they could penetrate, neither would exceed the other, as alſo the Lines AC, FL) will the Heights CM, LN, be equal.

I ſay, AC = FL.

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | ∠BAD = ∠KFI, AB : AC :: AC : AD. *Hyp.* |
| Alſo | 2 | FK = AC = FI. *Hyp.* |
| ∵ e. 1. 2. | 3 | ∠BAD = ∠KFI, AB : FK :: FI : AD. *Sub.* |
| ∵ e. 3. | 4 | BD = KI. 14. 6. |
| Again | 5 | BD : KI :: AE : FH. 32. |
| ∵ e. 5. | 6 | BD : AE :: KI : FH. 16. 5. |
| ∵ e. 4. 6. | 7 | BD : AE :: BD : FH. *Sub.* |
| ∵ e. 7. | 8 | AC = FL. 9. 5. Q. E. D. |

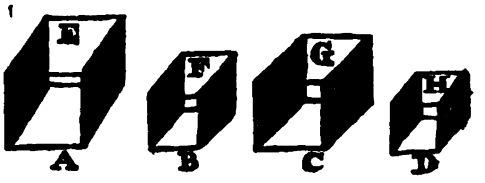
## PROP. XXXVII.

*If four Lines* A, B, C, D, *be proportional, that is, if* A : B :: C : D, *the Similar Parallelepipedons described upon those Lines are proportional. And if the Similar Parallelepipedons are proportional, those Lines shall be proportional.*

*Hyp.* 1 Suppose A : B :: C : D.

I say, 𝔄𝔈 : 𝔅𝔉 :: ℭ𝔊 : 𝔇ℌ.

### DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | A : B :: C : D. *Hyp.* |
| ∴ e. 1. | 2 | $A^1 : B^1 :: C^1 : D^1$. *Lem.* 2. |
| Also | 3 | $A^2 : B^2 :: AE : BF$. 33. |
| ∴ e. 2. 3. | 4 | $C^2 : D^2 :: AE : BF$. 11. 5. |
| Also | 5 | $C^3 : D^3 :: CG : DH$. 33. |
| ∴ e. 4. 5. | 6 | 𝔄𝔈 : 𝔅𝔉 :: ℭ𝔊 : 𝔇ℌ. 11. 5. Q. E. D. |

*Hyp.* 2 Suppose AE : BF :: CG : DH.

I say, 𝔄 : 𝔅 :: ℭ : 𝔇.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $AE : BF :: A^3 : B^3$. 33. |
| Also | 2 | AE : BF :: CG : DH. *Hyp.* |
| ∴ e. 1. 2. | 3 | $A^3 : B^3 :: CG : DH$. 11. 5. |
| Also | 4 | $C^3 : D^3 :: CG : DH$. 33. |
| ∴ e. 3. 4. | 5 | $A^3 : B^3 :: C^3 : D^3$. 11. 5. |
| ∴ e. 5. | 6 | 𝔄 : 𝔅 :: ℭ : 𝔇. Q. E. D. |

## PROP. XXXVIII.

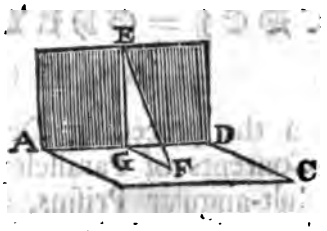
*If a Plane* ED, *be perpendicular to a Plane* AC, *and a Perpendicular* EF, *be drawn from a Point* E, *in one of the Planes* ED, *to the other Plane* AC, *that perpendicular* EF, *will fall on the common Section of the Planes* AD.

### PREPARATION.

If EF be ſaid to fall without AD, then in the Plane AC, let fall (12. 1.) FG ⊥ AD, and join EG; ſo (*Def.* 3.) will EFG, EGF, be Right-angles.

I ſay, **EF will fall on AD.**

### DEMONSTRATION.




For 1 If EF ſhould fall without AD.
Then 2 EFG, EGF, two ∠ˢ of a △, would be = 2 r∠.
But 3 Two ∠ˢ of a △ are ⊐ 2 r∠. 17. 1.
∴ 4 EF will not fall without AD.
∴ 5 **EF will fall on AD.** Q. E. D.

*I omit the* 39*th Propoſition becauſe of no great Conſequence.*

## PROP. XL.

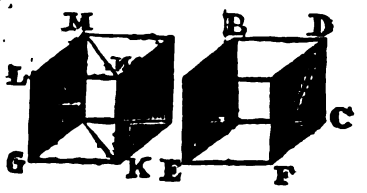
*A Prism* ABCDEF, *whose Base is a Parallelogram* AF, *double the Triangular Base* GHK, *of another Prism* GHKLMN, *of the same Altitude, is equal to that other Triangular Prism.*

### PREPARATION.

Compleat the Parallelepidons ED, HN.

I say, ABCDEF = GHKLMN.

### DEMONSTRATION.

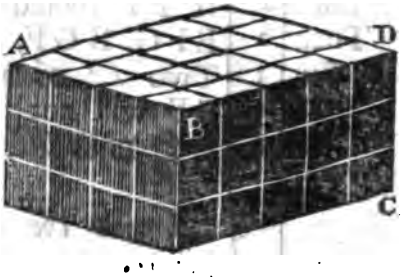



| | | |
|---|---|---|
| | 1 | AF = 2 GHK. *Hyp.* |
| Also | 2 | HK = 2 GHK. 41. 1. |
| ∴ e. 1. 2. | 3 | AF = HK. *Ax.* 7. |
| ∴ e. 3. | 4 | ED = HN. 31. |
| 4 ÷ 2 | 5 | ½ ED = ½ HN. *Ax.* 14. |
| Also | 6 | ½ ED = ABCDEF. 28. |
| ∴ e. 5. 6. | 7 | ½ HN = ABCDEF. *Ax.* 8. |
| Also | 8 | ½ HN = GHKLMN. 28. |
| ∴ e. 7. 8. | 9 | ABCDEF = GHKLMN. *Ax.* 8. Q. E. D. |

### SCHOLIUM.

From the preceding Demonstrations, the Solidities, or Solid Contents of Parallelepipedons, as also of Triangular, and Mult-angular Prisms, are learnt; *viz.* by multiplying their Bases into their Altitudes.

As if the Altitude CD, of a Right-angled Parallelepipedon AC, be 3 Inches, its Length BD 5, and its Breadth AB 4; then (*Lem.* 1. 1.) its Base AD = AB × BD = 4 × 5 = 20 Square Inches, which being multiplyed by its




Altitude

Altitude C D, or 3 Inches, will produce 60 Cubick Inches, for the Solidity of the Right-angled Parallelepipedon A C.

For as a Rect-angle, ſo alſo is a Right-Parallelepipedon A C, produced of the Baſe A D, drawn into the Altitude C D. Therefore every Parallelepipedon is produced of the Baſe drawn into the Altitude, ſince (31.) every Parallelepipedon is equal to a Right-angled one, conſtituted upon the ſame Baſe, and in the ſame Altitude.

Moreover, ſince the whole Parallelepipedon is produced of the Altitude drawn into the Baſe, the half thereof (that is, (28.) a Triangular Priſm) ſhall be produced of the Altitude drawn into half the Baſe.

'Tis the ſame alſo in Mult-angular Priſms, becauſe they may be reduced into Triangular Priſms.

*The End of the Eleventh Book.*

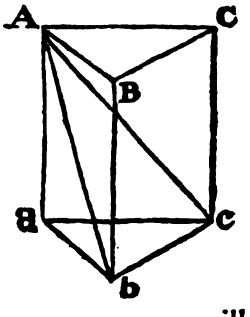
# EUCLID's ELEMENTS,

Demonſtrated after a New, Plain, and Eaſie METHOD.

## BOOK XII.

### DEFINITIONS.

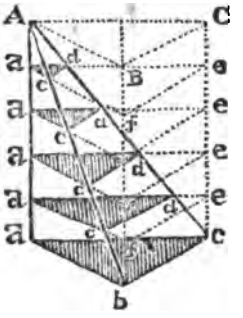
I. IF any Triangle A B C, be conceiv'd to move from the Vertex or Top of a Solid Angle, (determin'd by two Planes A *a b*, A *a c*, joined together in the common Line A *a*,) with a Motion always parallel to its ſelf; ſo that its extream Angular Point A, ſhall always remain in the Line A *a*, but with its Sides A B, A C, ſhall all along raze on the Triangular Planes A *a b*, A *a c*, 'till at length they fall wholly within the Solid Angle, by this its Motion it will deſcribe within the Solid Angle, the Figure called a **Pyramid**, whoſe Baſe will be the Triangular *a b c*, and its Vertex A, and it




will

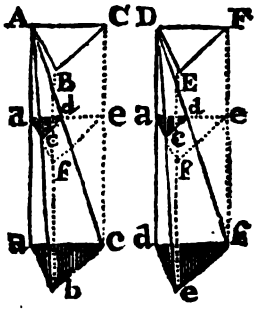
will alſo deſcribe without it, another Quadrangular Pyramid, whoſe common Vertex will be the ſame A, but the Baſe the Quadrangle C b.

The firſt Pyramid it will deſcribe with its Triangular Parts a c d, continually increaſing from the Point A, and ending in the Triangle a b c, but the latter Pyramid will be deſcribed by the remaining Parts, or Trapezia c d e f, continually decreaſing downwards from the whole A B C, and at Length ending in the Right-Line b c; ſo that in the mean while, the ſaid Triangle with its whole Space, deſcribes the Triangular Priſm, compos'd of thoſe two Pyramids.




## COROLLARY I.

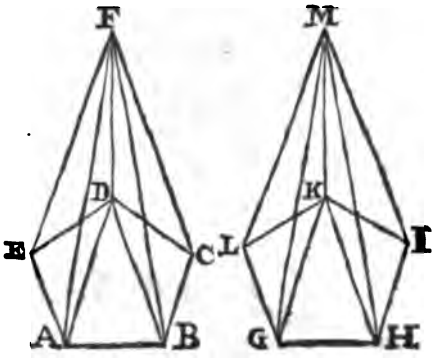
Of what Sort ſoever the Deſcribing Triangles A B C, D E F, are, ſo they are equal; and whatever the Solid Angles are, comprehended under the Planes A a b, A a c, and D d e, D d f, ſo they are accommodated to the Plane Angles A and D, and ſuch that the Deſcribing Triangles A B C, D E F, conceiv'd to be mov'd within them, thro' an equal Interval of Motion from the Line A C to a c, and from D F to d f, may be all (or wholly) receiv'd within them in the ſame Moment of Time; becauſe the increaſing Triangular Planes a c d, in each Triangular Pyramid, as alſo the decreaſing Trapezia c d e f, in each Quadrangular Pyramid, muſt neceſſarily be conceiv'd to anſwer one the other, both in Number and Magnitude, therefore all their Sums on both Sides will be equal, and conſequently there will be deſcribed the equal Triangular Pyramids a b c A, d e f D, within the Solid Angles and the equal Quadrangular Ones b c C B A, e f F E D, without the Solid Angles; and ſo Triangular and Quadrangular Pyramids, of equal Baſes and Altitudes, may be undoubtedly taken for equal ones.



## COROLLARY II.

Therefore all Pyramids, of equal Baſes and Altitudes, are equal the one to the other.

This (*Cor.* 1.) is evident of Triangular and Quadrangular Pyramids.

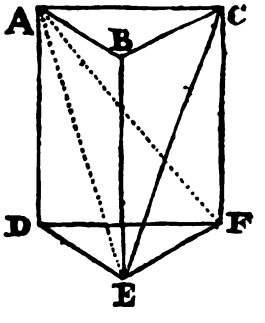



It is alſo as certain of Polygonal Pyramids ABCDEF, GHIKLM, becauſe they may be reſolved into Trianlar Pyramids; which will anſwer one another both in Number and Magnitude, in each Polygonal Pyramid.

## COROLLARY III.

In the Triangular Priſm A B C D E F, the Quadrangular Pyramid B C F E A, is preciſely Double of the other Triangular Pyramid DEFA; and the Triangular Priſm ABCDEF, is Treble of the Triangular Pyramid D E F A.

For having drawn the Diagonal E C, the Quadrangular Pyramid BCFEA, will be divided into two Triangular Pyramids EBCA, EFCA, which will have a common Vertex A, and equal Triangular Baſes, and conſequently (*Cor.* 2.) will be equal.



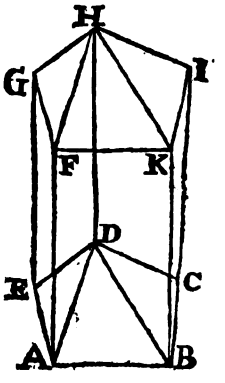

But one of theſe E B C A, or A B C E, (*viz.* if we take A B C for the Baſe, and E for the Vertex,) is (*Cor.* 2.) equal to DEFA. Therefore the Quadrangular Pyramid B C F E A, is = 2 DEFA, and conſequently the Triangular Priſm A B C D E F, is = 3 D E F A.

CO-

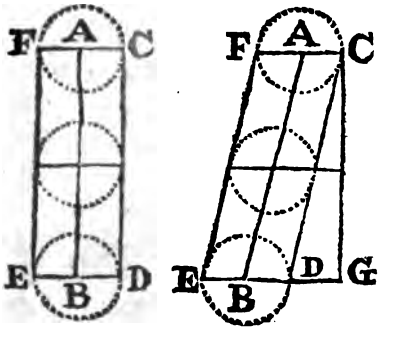
## COROLLARY IV.

Hence, every Priſm is Treble of the Pyramid, that has the ſame Baſe and Height with it; or every Pyramid is the third Part of the Priſm, that has the ſame Baſe and Height with it.

For reſolve the Polygonal Priſm ABCDEGFKIH, into Triangular Priſms BCDHKI, BDAFKH, ADEGFH; and the Pyramid ABCDEG, into Triangular Pyramids BCDG, BDAG, ADEG; ſo (*Cor.* 3.) will BCDHKI be = 3 BCDG, BDAFKH = 3 BDAG, and ADEGFH = 3 ADEG, that is, all the Parts of the Priſm will be Treble to all the Parts of the Pyramid, and conſequently the whole Priſm ABCDEGFKIH, is Treble the whole Pyramid ABCDEG.




2 If a Circle A, be conceiv'd to move downwards, along a Right-line CD, with a Motion always parallel to it ſelf, by this its Motion it will generate a Solid CDEF, called a **Cylinder**, which, if the Right-line CD be perpendicular, is called a **Right-Cylinder**, if oblique, an **Inclined-Cylinder**.



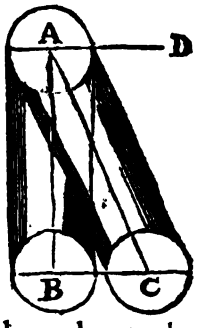
## COROLLARY I.

From this Geneſis of Cylinders it is evident, that their Solid Contents may be found, by multiplying their Baſes into their perpendicular Altitudes; becauſe they muſt neceſſarily be conceiv'd to conſiſt of as many Elements, or Circles, parallel and equal to the Deſcribing Planes, or Baſes A, as there are Points, or equal Parts, indefinitely many and ſmall, in their perpendicular Altitudes A B, C G.

## COROLLARY II.

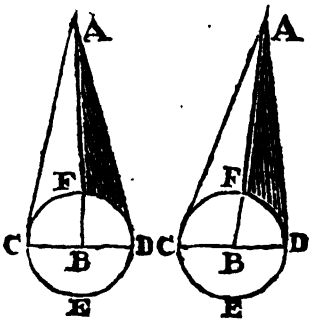
The Solidities therefore, both of Right and Inclined Cylinders, being found by multiplying their Baſes A, into their perpendicular Altitudes A B, C G; if their Baſes A, and Altitudes A B, C G, be equal, their Solid Contents, or the Cylinders themſelves, will be alſo equal.

Hence alſo it is evident, that Cylinders A B, A C, which are conſtituted upon the ſame Baſe A, and in the ſame Altitude A B, that is, within the ſame Parallels A D, B C, are equal.




3 **The Axes of Cylinders**, are thoſe Right-lines A B, A C, which join the Centers of their Baſes.

4 If a Triangle A B C, be conceiv'd to revolve about the immoveable Side A B, till the Revolution be intire, that is, till the Side B C, arrive at the Place where it began; by this its Motion it will deſcribe a Solid A C E D F, called a **Cone**, which, if generated by the Motion of a Right-angled Triangle, is called a **Right-Cone**, if by that of an Oblique angled Triangle, an **Inclined-Cone**.

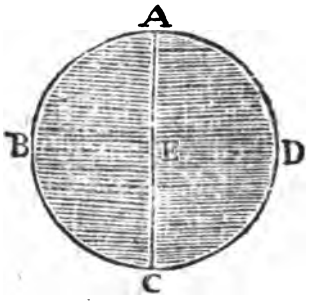



The

5 **The Axis of a Cone**, is that Right-line AB, which joins its Vertex A, and the Center B, of its Circular Base.

6 Right-Cylinders and Cones are **Similar**, when their Axes are proportional to the Diameters of their Bases; but Inclined Cylinders and Cones are not **Similar**, unless they have a third Condition, *viz.* that their Axes be also equally inclined to the Planes of their Bases.

7 If a Semicircle A B C, be conceiv'd to move round the immoveable Diameter A C, till its Circumference A B C, come to the Place where it began to move, by this its Motion, it will describe a Solid ABCD, called a **Sphere**.



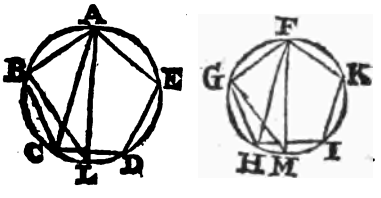

8 **The Axis of a Sphere**, is that immoveable Line A C, about which the Semicircle is turned.

9 **The Center of a Sphere**, is that Point from which all Right-lines drawn to the Surface, are equal; as E.

10 **The Diameter of a Sphere**, is any Right-line whatsoever, which passes thro' its Center, and is bounded on each Side by its Surface.

## PROPOSITION I.

*Similar Polygonal Figures* AB CDE, FGHIK, *inscribed in Circles* ABD, FGI, *are one to another, as the Squares described of the Diameters of the Circles* A L, F M.



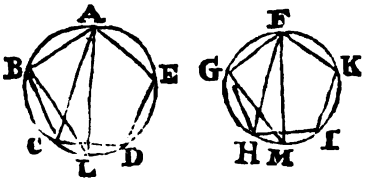
## PREPARATION.

Draw AC, BL, FH, GM.

I say, $\mathfrak{A}\mathfrak{L}q : \mathfrak{F}\mathfrak{M}q :: \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}\mathfrak{E} : \mathfrak{F}\mathfrak{G}\mathfrak{H}\mathfrak{J}\mathfrak{K}$.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $\angle ABC = \angle FGH$; AB : BC :: FG : GH. *Def.* 1. 6. |
| ∴ e. 1. | 2 | $\angle ACB = \angle FHG$. 6. 6. |
| Alſo | 3 | $\angle ACB = \angle ALB$. 21. 3. |
| ∴ e. 2. 3. | 4 | $\angle FHG = \angle ALB$. *Ax.* 1. |
| Alſo | 5 | $\angle FHG = \angle FMG$. 21. 3. |
| ∴ e. 4. 5. | 6 | $\angle ALB = \angle FMG$. *Ax.* 1. |
| Again | 7 | $\angle ABL = \angle FGM$. 31. 3. |
| ∴ e. 6. 7. | 8 | $\angle ALB = \angle FMG$, $\angle ABL = \angle FGM$. |
| ∴ e. 8. | 9 | $\angle BAL = \angle GFM$. *Cor.* 2. 32. 1. |
| ∴ e. 8. 9. | 10 | $\angle ALB = \angle FMG$, $\angle ABL = \angle FGM$, $\angle BAL = \angle GFM$. |
| ∴ e. 10. | 11 | AB : FG :: AL : FM. *Scho.* 3. 4. 6. |
| ∴ e. 11 | 12 | $ABq : FGq :: ALq : FMq$. *Scho. Lem.* 2. 11. |
| Alſo | 13 | $ABq : FGq ::$ ABCDE : FGHIK. 20. 6. |
| ∴ e. 12. 13. | 14 | $\mathfrak{A}\mathfrak{L}q : \mathfrak{F}\mathfrak{M}q :: \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}\mathfrak{E} : \mathfrak{F}\mathfrak{G}\mathfrak{H}\mathfrak{J}\mathfrak{K}$. 11. 5. Q. E. D. |




## LEMMA I.

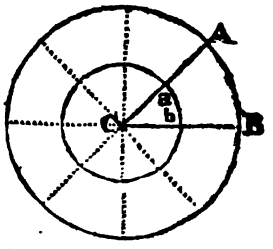
The Peripheries of Circles are in Proportion one to another, as their Radius's.

## PREPARATION.

Having (by Radius's drawn from the Centers) divided the Periphery of a Circle, into any Number of equal Arches, ſuppoſe 8, whereof AB is one; thro' any Point in the Radius, draw a Concentric or Parallel Circle; ſo (*Cor.* 3. 33.

3. 33. 6.) will its Periphery be alſo divided into 8 equal Arches, by thoſe Radius's, one of which will be a b. Moreover the Periphery of one Circle will be 8 a b, that of the other 8 A B, and (∠ C being common, ∠ a (29. 1.) = ∠ A, and ∠ b = ∠ B;) the Triangles C a b, C A B, will be Equiangular.

I ſay, $\mathfrak{r} : \mathfrak{R} :: \mathfrak{p} : \mathfrak{P}$.




## DEMONSTRATION.

1 C a : C A :: a b : A B. *Scho.* 3. 4. 6.
∵ e. 1. 2 C a : C A :: 8 a b : 8 A B.
i. e. 3 $\mathfrak{r} : \mathfrak{R} :: \mathfrak{p} : \mathfrak{P}$. Q. E. D.

## COROLLARY.

Hence, if the Radius, or Semidiameter of one Circle, be called X, that of another Y, their Peripheries p, P, will have the ſame Proportion to one another, as X Z to Y Z.

For 1 X : Y :: p : P. *Lem.* 1.
Alſo 2 X : Y :: X Z : Y Z. *Lem.* 2. 5.
∵ e. 1. 2. 3 $\mathfrak{p} : \mathfrak{P} :: \mathfrak{X}\mathfrak{Z} : \mathfrak{Y}\mathfrak{Z}$. 11. 5.

## LEMMA II.

The Sum of a Series of Quantities, proceeding from o (incluſive) in an Arithmetical Progreſſion, or whoſe common Difference is equal to the leaſt Quantity; as o, A, 2 A, 3 A, 4 A, 5 A, &c. is Subduple of the Product of the laſt or greateſt Term, and Number of Terms.

PRE-

## PREPARATION.

Let the Sum of the Series be called S, the laſt or greateſt Term G, and the Number of Terms N.

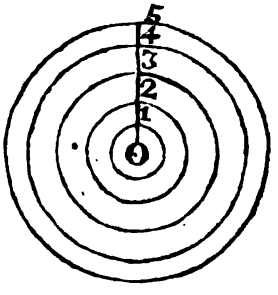
I ſay, $S = \frac{NG}{2}$.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $G = \overline{N-1} \times A$. |
| $1 \times N$ | 2 | $NG = \overline{N-1} \times NA$. *Ax.* 13. |
| $2 \div \overline{2}$ | 3 | $\frac{NG}{2} = \frac{\overline{N-1} \times NA}{2}$. *Ax.* 14. |
| Alſo | 4 | $S = \frac{\overline{N-1} \times NA}{2}$. |
| ∴ c. 4. 3. | 5 | $S = \frac{NG}{2}$. *Ax.* 1. Q. E. D. |

## COROLLARIES.

1 Hence, if a Circle be conceiv'd to conſiſt of as many Concentric Peripheries, as there are Points, or equal Parts indefinitely many and ſmall in the Radius; theſe Peripheries, as well as their Radius's, will (*Lem.* 1.) proceed from the Center, or o, in an Arithmetical Progreſſion; and therefore their Sum, that is, the whole Circle, is equal to half the greateſt (or extream Circumference) drawn into the Number of Terms, that is, the Radius.




Where-

2 Wherefore if the Radius, or Semidiameter of any Circle, be called X, that of another Y, their Peripheries or Circumferences will (*Cor. Lem.* 1.) be to one another, as XZ to YZ, and their Areas, as $\frac{1}{2}$ XZX to $\frac{1}{2}$ YZY.

## PROP. II.

*Circles are to one another, as the Squares of their Diameters.*

### PREPARATION.

Suppose the Radius, or Semidiameter of one Circle = X, that of another = Y; so will their Diameters be 2X and 2Y, and the Squares of their Diameters 4XX and 4YY; also the Area of one will (*Cor.* 2. *Lem.* 2.) be $\frac{1}{2}$ XZX, that of the other $\frac{1}{2}$ YZY.

I say, $4\mathfrak{X}\mathfrak{X} : 4\mathfrak{Y}\mathfrak{Y} :: \frac{1}{2}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{2}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y}$.

### DEMONSTRATION.

1 $4XX \times \frac{1}{2}YZY = 4YY \times \frac{1}{2}XZX$.

2 $4\mathfrak{X}\mathfrak{X} : 4\mathfrak{Y}\mathfrak{Y} :: \frac{1}{2}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{2}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y}$. *Lem.* 2. 5. Q. E. D.

i. e. 1.

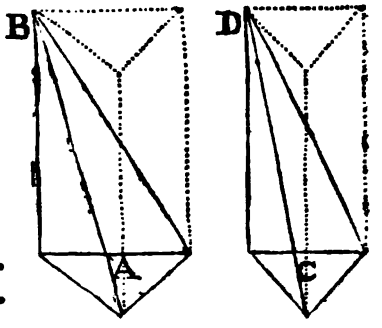
*The third and fourth Propositions are needless; because they only serve to demonstrate the fifth and sixth, that I shall demonstrate otherwise, and more easily.*

## PROP. V.

*Triangular Pyramids* AB, CD, *of the ſame Height, are in Proportion one to another, as their Baſes* A, C.

### PREPARATION.

Erect upon the Baſes A, C, of the Triangular Pyramids AB, CD, Triangular Priſms of the ſame Height with them; ſo (*Cor.* 3. *Def.* 1.) will thoſe Priſms be among themſelves, as 3 AB to 3 CD.




I ſay, A : C :: AB : CD.

### DEMONSTRATION.

1 A : C :: 3 AB : 3 CD. *Cor.* 32. 11.
Alſo 2 AB : CD :: 3 AB : 3 CD. *Lem.* 2. 5.
Vo. 1. 2. 3 A : C :: AB : CD. 11. 5. Q. E. D.

## PROP. VI.

*All Sorts of Pyramids, of the ſame Height, have the ſame Proportion, as their Baſes.*

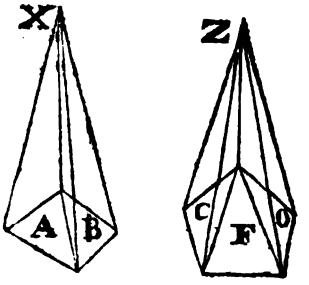
### PREPARATION.

Let the Pyramids be ABX, OFCZ, and let their Baſes AB, OFC, be reſolved into the Triangles A, B, C, F, O, and the Pyramids themſelves into the Triangular Pyramids AX, BX, CZ, FZ, OZ.

I ſay, AB : OFC :: ABX : OFCZ.

DE-

## DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | A X : O Z :: A : O ; B X : O Z :: B : O. 5. |
| ∴ e. 1. | 2 | A X + B X : O Z :: A + B : O 24. 5. |
| ∴ e. 2. | 3 | A X + B X : A + B :: O Z : O. 16. 5. |
| i. e. | 4 | A B X : A B :: O Z : O. |
| Again | 5 | A X : F Z :: A : F ; B X : F Z :: B : F. 5. |
| ∴ e. 5. | 6 | A X + B X : F Z :: A + B : F. 24. 5. |
| ∴ e. 6. | 7 | A X + B X : A + B :: F Z : F. 16. 5. |
| i. e. | 8 | A B X : A B :: F Z : F. |
| Laſtly | 9 | A X : C Z :: A : C ; B X : C Z :: B : C. 5. |
| ∴ e. 9. | 10 | A X + B X : C Z :: A + B : C. 24. 5. |
| ∴ e. 10. | 11 | A X + B X : A + B :: C Z : C. 16. 5. |
| i. e. | 12 | A B X : A B :: C Z : C. |
| ∴ e. 4. 8. 12. | 13 | A B X : A B :: O Z : O :: F Z : F :: C Z : C. 11. 5. |
| ∴ e. 13. | 14 | A B X : A B :: O Z + F Z + C Z : O + F + C. 12. 5. |
| ∴ e. 14. | 15 | A B : O + F + C :: A B X : O Z + F Z + C Z. *Cor.* 7. 5. |
| i. e. | 16 | 𝔄 𝔅 : 𝔇 𝔉 ℭ :: 𝔄 𝔅 𝔛 : 𝔇 𝔉 ℭ ℨ. Q. E D. |

## COROLLARIES.

1. Hence, All Sorts of Pyramids, of equal Baſes and Altitudes, are equal.

2. Becauſe (*Cor.* 4. *Def.* 1.) every Priſm is Treble of the Pyramid, that has the ſame Baſe and Height with it; therefore all Sorts of Priſms of the ſame Height, have the ſame Proportion, as their Baſes.

## PROP. VII.

*Every Prism is Treble of the Pyramid, that has the same Base and Height with it; or every Pyramid is the third Part of the Prism, that has the same Base and Height with it.*

This is Demonstrated *Cor.* 4. *Def.* 1.

## LEMMA III.

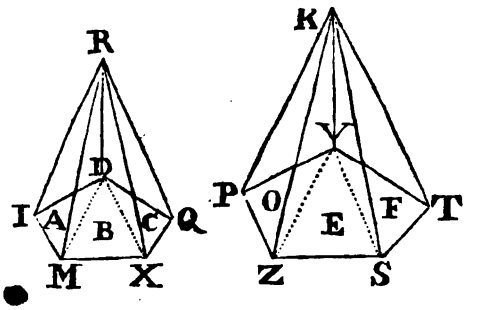
Similar Polygonal Pyramids ABCR, OEFK, may be resolved into the Similar Triangular Pyramids AR and OK, BR and EK, CR and FK.

## PREPARATION.

Draw MD, XD, ZV, SV.

I say, AR and OK, BR and EK, CR and FK, are Similar.

## DEMONSTRATION.




1 | $\triangle^{s}$ A, O, are Similar. 20. 6.
e. 1. 2 | MD : MI :: ZV : ZP. *Def.* 1. 6.
Also 3 | $\triangle^{s}$ MIR, ZPK, are Similar. *Def.* 12. 11.

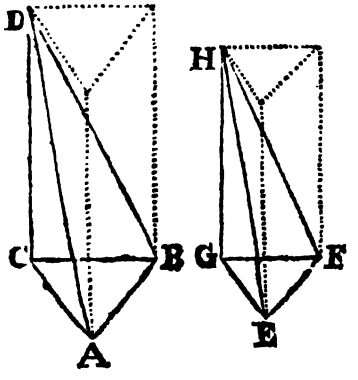
MI :

| | | |
|---|---|---|
| ∴ e. 3. | 4 | MI : MR :: ZP : ZK. *Def.* 1. 6. |
| ∴ e. 2. 4. | 5 | MD : MI :: ZV : ZP ; MI : MR :: ZP : ZK. |
| ∴ e. 5. | 6 | MD : MR :: ZV : ZK. 22. 5. |
| ∴ e. 6. | 7 | MR : MD :: ZK : ZV. *Cor.* 7. 5. |
| Again | 8 | After the ſame manner, MD : RD :: ZV : KV. |
| ∴ e. 7. 8. | 9 | MR : MD :: ZK : ZV ; MD : RD :: ZV : KV. |
| ∴ e. 9. | 10 | MR : RD :: ZK : KV. 22. 5. |
| ∴ e. 6. 8. 10. | 11 | MD : MR :: ZV : ZK ; MD : RD :: ZV : KV ; MR : RD :: ZK : KV. |
| ∴ e. 11. | 12 | △ MRD, ZKV, are Equiangular. 5. 6. |
| ∴ e. 12. | 13 | △ MRD, ZKV, are Similar. *Scho.* 1. 4. 6. |
| ∴ e. 13. | 14 | AR, OK, are Similar. *Def.* 12. 11. |
| Laſtly | 15 | After the ſame manner, △ XRD, SKV, are Similar. |
| ∴ e. 15. | 16 | BR and EK, CR and FK, are Similar. *Def.* 12. 11. |
| ∴ e. 14. 16. | 17 | **AR and DK, BR and EK, CR and FK, are Similar.** Q. E. D. |

## PROP. VIII.

*Similar Pyramids are in the Triplicate Ratio, or as the Cubes of their Homologous Sides.*

*Caſe* 1 If the Pyramids ABCD, EFGH, be Triangular.



2 The Homologous Sides IM, PZ, of Similar Pyramids ABCR, OEFK, being therefore given; if IM, PZ, G, H, be continual Proportionals, it will be IM:H::ABCR:OEFK.

3 Becauſe (7.) every Priſm is Treble of the Pyramid, that has the ſame Baſe and Height with it; therefore Similar Priſms are in the Triplicate Ratio, or as the Cubes of their Homologous Sides.

## PROP. IX.

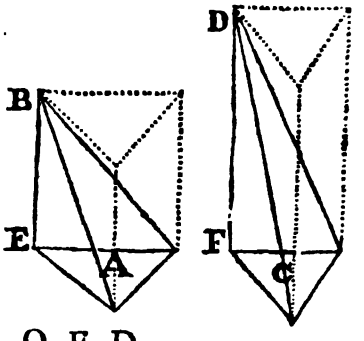
*Equal Pyramids have their Baſes and Altitudes reciprocal: And thoſe Pyramids which have their Baſes and Altitudes reciprocal, are equal.*

*Hyp.* 1 Suppoſe the Pyramids to be equal.

*Caſe* 1 If the Pyramids AB, CD, be Triangular.

## PREPARATION.

Erect upon their Baſes A, C, Triangular Priſms, of the ſame Heights EB, FD, with them; ſo (they being (7.) Triple of the equal Pyramids) will thoſe Priſms be alſo equal.




I ſay, 𝔄:ℭ::𝔉𝔇:ℭ𝔅.

DEMONSTRATION.

1 𝔄:ℭ::𝔉𝔇:ℭ𝔅. *Cor.* 34. 11. Q. E. D.

*Caſe* 2 If the equal Pyramids be Polygonal; their Baſes and Altitudes will be reciprocal.

DE.

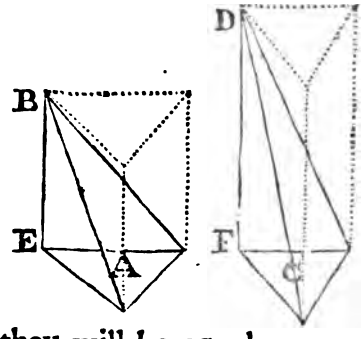
DEMONSTRATION.

Imagine the given Polygonal Baſes, to be (25. 6.) reſolved into Triangular Baſes, equal to them; and upon thoſe Triangular Baſes, imagine Triangular Pyramids to be erected, of the ſame Heights with the given Polygonal Pyramids; ſo (*Cor.* 1. 6.) will they be equal to the equal Polygonal Pyramids, and conſequently to one another. Since therefore thoſe equal Triangular Pyramids, have (*Caſe* 1.) their Baſes and Altitudes reciprocal, the given Polygonal Pyramids, their Equals, will alſo have their Baſes and Altitudes reciprocal. Q. E. D.

*Hyp.* 2 Suppoſe the Pyramids to have their Baſes and Altitudes reciprocal

*Caſe* 1 If the Pyramids AB, CD, be Triangular.

Erect upon their Baſes A, C, Triangular Priſms, of the ſame Heights EB, FD, with them; ſo (7.) will thoſe Priſms be among themſelves, as 3 AB to 3 CD.




I ſay, $\mathfrak{AB} = \mathfrak{CD}$.

1 3 AB = 3 CD. *Cor.* 34. 11.

1 ÷ 3 2 $\mathfrak{AB} = \mathfrak{CD}$. *Ax.* 14. Q. E. D.

*Caſe* 2 If the Pyramids be Polygonal; they will be equal.

DEMONSTRATION.

Imagine the given Polygonal Baſes, to be (25. 6.) reſolved into Triangular Baſes, equal to them; and upon thoſe Triangular Baſes, imagine Triangular Pyramids to be erected, of the ſame Heights with the given Polygonal Pyramids; ſo (*Cor.* 1. 6.) will they be equal to the given Polygonal Pyramids, and have their Baſes and Altitudes reciprocal. Since therefore thoſe Triangular Pyramids are (*Caſe* 1.) equal, the given Polygonal Pyramids, their Equals, will be alſo equal. Q. E. D.

## COROLLARY.

Becauſe (7.) every Priſm is Treble of the Pyramid, that has the ſame Baſe and Height with it; therefore equal Priſms have their Baſes and Altitudes reciprocal: And thoſe Priſms which have their Baſes and Altitudes reciprocal, are equal.

## LEMMA IV.

If the firſt Term of never ſo many continual Proportionals, A, AR, ARR, ARRR, &c. *ad Infinitum*, be ſubtracted from the laſt, and the Remainder divided by the Ratio leſſened by Unity, the Quotient will be equal to the Sum of all the Terms except the laſt.

## PREPARATION.

Let the Number of Terms be called N, the laſt Term L, and the Sum of all the Terms except the laſt S; ſo will S be $= A + AR + ARR = \frac{AR^{n-1} - A}{R-1}$; as will appear by multiplying $R - 1$ into $A + AR + ARR$.

I ſay, $\frac{L-A}{R-1} = S$.

## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $L = AR^{n-1}$. |
| $1 - A$ | 2 | $L - A = AR^{n-1} - A$. *Ax.* 12. |
| $\div R-1$ | 3 | $\frac{L-A}{R-1} = \frac{AR^{n-1} - A}{R-1}$. *Ax.* 14. |
| Alſo | 4 | $S = \frac{AR^{n-1} - A}{R-1}$. |
| 3. 4. | 5 | $\frac{L-A}{R-1} = S$. *Ax.* 1. Q. E. D. |

COR.

## COROLLARIES.

1 In a continued Series of Fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. decreasing in a Duple Proportion *ad Infinitum*, if you take them backwards, you may justly reckon a Cypher, or 0, for the first Term, (for between $\frac{1}{2}$ and 0, there may be an infinite Number of such Terms,) and the Sum of all the Terms except the last will be $= \frac{1}{2}$; for subtracting the first 0, from the last $\frac{1}{2}$, and dividing the Remainder $\frac{1}{2}$, by the Ratio lessened by Unity, that is, by 1, which divides Nothing, the Quotient $\frac{1}{2}$, is (*Lem.* 4.) the Sum of all the Terms, except the last.

2 The Sum of $\frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4}$, &c. decreasing in a Quadruple Proportion *ad Infinitum*, will be $\frac{1}{4^1}$; for subtracting the first Term 0, from the last $\frac{1}{4^2}$, and dividing the Remainder $\frac{1}{4^2}$, by the Ratio lessened by Unity, that is, by 3, the Quotient $\frac{1}{4^2}$ will (*Lem.* 4.) give the Sum of all the Terms except the last. This therefore (*viz.* $\frac{1}{4^1}$) being added, the Sum of all will be $= \frac{1}{4^1} = \frac{1}{3}$.

3 Wherefore $\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$, &c. *ad infinitum*, or $-\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, &c. *ad Infinitum*, will (*Cor.* 1.) be $= 0$; also $\frac{1}{16} - \frac{1}{64} - \frac{1}{256} - \frac{1}{1024}$, &c. *ad Infinitum*, or $-\frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024}$, &c. *ad Infinitum*, will (*Cor.* 2.) be $= 0$.

## LEMMA V.

The Sum of a Series of Quantities, proceeding in a Duplicate Arithmetick Progression from 1, *ad Infinitum*, that is, 1, 4, 9, 16, &c. the Squares of Numbers in an Arithmetick Progression 1, 2, 3, 4, &c. is Subtriple of the Sum of as many Terms equal to the greatest, as is the Number of Terms.

## DEMONSTRATION.

The Sum of three Terms, 1, 4, 9, = 14, is to thrice 9 = 27, (dividing both Sides by 9) as $1\frac{5}{9}$, or $1\frac{10}{18}$, or $1 + \frac{1}{2} + \frac{1}{18}$ to 3.

The Sum of Six Terms 1, 4, 9, 16, 25, 36, = 91, is to Six Times 36 = 216, (dividing both Sides by 72) as $1 + \frac{1}{4} + \frac{1}{72}$, to 3.

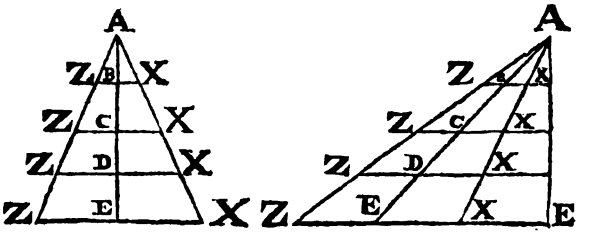
The Sum of twelve Terms 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, = 650, is to Twelve Times 144 = 1728 (dividing both Sides by 576) as $1 + \frac{1}{2} + \frac{1}{288}$ to 3.

The Fractions adhering to them thus conſtantly decreaſing, ſome by their half Parts, others by three Quarters, (for $\frac{2}{18}$ is $\frac{4}{72}$, therefore the firſt Decrement is $\frac{3}{72}$; and $\frac{1}{72}$ is $\frac{4}{288}$, therefore the ſecond Decrement is $\frac{1}{288}$;) the Sum of an infinite Series, will be to the Sum of as many Terms equal to the greateſt,

as $1 + \frac{1}{2} + \frac{1}{18}$
$- \frac{1}{4} - \frac{3}{72}$
$- \frac{1}{8} - \frac{3}{288}$
$- \frac{1}{16}$, &c. $- \frac{1}{1152}$, &c. to 3, that is, (*Cor.* 3. *Lem.* 4.) as 1 to 3. Q. E. D.

CO-

## COROLLARY.




Hence, if we ſuppoſe the Axis AE, either of a Right, or Inclined Cone AZX, to be divided into equal and indefinitely many Parts, by as many parallel Right-lines ZX, the Lines ZX, will (it being (*Scho.* 3. 4. 6.) as AB:AC::BZ:CZ::BX:CX, and AD:AE::DZ:EZ::DX:EX.) be as the Numbers 1, 2, 3, 4, *&c.* and their Squares, or (2.) the Circles conſtituted on the Diameters ZX, as 1, 4, 9, 16, *&c.*

Whence all thoſe Circles, or the whole Cone AZX, (made up of the ſame) will be Subtriple of as many Circles equal to the Greateſt, conſtituted on the greateſt Diameter ZEX, that is, Subtriple of a Cylinder whoſe Baſe is ZEX, and Axis or Altitude AE.

## PROP. X.

*Every Cone is the third Part of a Cylinder of the ſame Baſe and Altitude.*

*This is Demonſtrated,* Cor. Lem. 5.

## COROLLARY.

Becauſe the Solid Contents of Cylinders, may (*Cor.* 1. *Def.* 2.) be learnt, by multiplying their Baſes into their perpendicular Altitudes, and every Cone is the third Part of a Cylinder, of the ſame Baſe and Altitude; therefore the Solid Contents of Cones, may be learnt, by multiplying a third Part of their Baſes into their perpendicular Altitudes.

## PROP. XI.

*Cylinders and Cones, of the ſame Altitude, are as their Baſes.*

## PREPARATION.

Let the Circular Baſes, of the Cylinders and Cones, be (*Cor.* 2. *Lem.* 2.) expreſſed by $\frac{1}{2}XZX$, and $\frac{1}{2}YZY$, and their equal Altitudes by **A**; ſo (*Cor.* 1. *Def.* 2.) will the Cylinders be among themſelves, as $\frac{1}{2}AXZX$ to $\frac{1}{2}AYZY$, and (*Cor.* 10.) the Cones, as $\frac{1}{6}AXZX$ to $\frac{1}{6}AYZY$.

I ſay, $\frac{1}{2}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{2}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y} :: \frac{1}{2}\mathfrak{A}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{2}\mathfrak{A}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y} :: \frac{1}{6}\mathfrak{A}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{6}\mathfrak{A}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y}$.

## DEMONSTRATION.

1 $\frac{1}{2}XZX : \frac{1}{2}YZY :: \frac{1}{2}AXZX : \frac{1}{2}AYZY$. *Lem.* 2. 5.

Alſo 2 $\frac{1}{2}XZX : \frac{1}{2}YZY :: \frac{1}{6}AXZX : \frac{1}{6}AYZY$. *Lem.* 2. 5.

∴ e. 1. 2. 3 $\frac{1}{2}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{2}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y} :: \frac{1}{2}\mathfrak{A}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{2}\mathfrak{A}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y} :: \frac{1}{6}\mathfrak{A}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{6}\mathfrak{A}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y}$. 11. 5. Q. E. D.

LEM

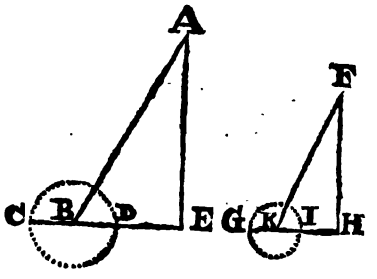
## LEMMA VI.

*Similar Cylinders and Cones, have their Altitudes proportional to the Diameters of their Bases.*

## PREPARATION.

If the Cylinders and Cones are Right, their Axes or Altitudes, will (*Def.* 6.) be proportional to the Diameters of their Bases; if Inclined, from the Points A, F, of their Axes, AB, FK, upon the Diameters of their Bases CD, GI, produced, let fall the Perpendiculars AE, FH, so ($\angle AEB$ being (*Ax.* 21.) $= \angle FHK$, $\angle ABE$ (*Def.* 6.) $= \angle FKH$) will $\angle BAE$ be (*Cor.* 2. 32. 1.) $= \angle KFH$ and consequently the Triangles ABE, FKH, will be Equiangular.

I say, CD : GI :: AE : FH.




## DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | AB : FK :: CD : GI. *Def.* 6. |
| Also | 2 | AB : FK :: AE : FH. *Scho.* 3. 4. 6. |
| ∴ e. 1. 2. | 3 | CD : GI :: AE : FH. 11 5. Q. E. D. |

## PROP. XII.

*Similar Cylinders and Cones, are in the Triplicate Ratio, or as the Cubes of the Diameters of their Baſes.*

### PREPARATION.

The Semi-diameters of the Circular Baſes, of the Similar Cylinders and Cones, being (*Cor.* 2. *Lem.* 2.) expreſſed by X and Y, and their Baſes by $\frac{1}{3}$XZX and $\frac{1}{3}$YZY, let their Altitudes be repreſented by A and *a*; ſo (*Cor.* 1. *Def.* 2.) will the Cylinders be among themſelves as $\frac{1}{3}$AXZX to $\frac{1}{3}a$YZY, and (*Cor.* 10.) the Cones as $\frac{1}{3}$AXZX to $\frac{1}{3}a$YZY. Moreover the Diameters of their Baſes being 2X and 2Y, their Cubes, will be 8XXX and 8YYY.

I ſay, $8\mathfrak{X}' : 8\mathfrak{Y}' :: \frac{1}{3}\mathfrak{A}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{3}\mathfrak{a}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y} :: \frac{1}{3}\mathfrak{A}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{3}\mathfrak{a}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y}$.

### DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | $2X : 2Y :: A : a$. *Lem.* 6. |
| ∵ e. 1. | 2 | $2YA = 2Xa$. *Lem.* 1. 5. |
| $2 \div 2$ | 3 | $YA = Xa$. *Ax.* 14. |
| $3 \div Y$ | 4 | $A = \frac{Xa}{Y}$. *Ax.* 14. |
| $4 \times \frac{1}{3}XZX$ | 5 | $\frac{1}{3}AXZX = \frac{\frac{1}{3}aXZXX}{Y}$. *Ax.* 13. |
| Again | 6 | $8X' : 8Y' :: \frac{\frac{1}{3}aXZXX}{Y} : \frac{1}{3}aYZY$. *Lem.* 2. 5. |
| ∵ e. 5. 6. | 7 | $8X' : 8Y' :: \frac{1}{3}AXZX : \frac{1}{3}aYZY$. *Sub.* |
| Alſo | 8 | $\frac{1}{3}AXZX : \frac{1}{3}aYZY :: \frac{1}{3}AXZX : \frac{1}{3}aYZY$. *Lem.* 2. 5. |
| ∴ e. 7. 8. | 9 | $8\mathfrak{X}' : 8\mathfrak{Y}' :: \frac{1}{3}\mathfrak{A}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{3}\mathfrak{a}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y} :: \frac{1}{3}\mathfrak{A}\mathfrak{X}\mathfrak{Z}\mathfrak{X} : \frac{1}{3}\mathfrak{a}\mathfrak{Y}\mathfrak{Z}\mathfrak{Y}$. 11 5. Q. E. D. |

COROL.

## COROLLARIES.

1. Becauſe the Axes of Similar Cylinders and Cones, are (*Def.* 6.) in the ſame Ratio, as the Diameters of their Baſes; therefore Similar Cylinders and Cones, are in the Triplicate Ratio, or as the Cubes of their Axes.

2. Becauſe the Altitudes of Similar Cylinders and Cones, are (*Lem.* 6.) in the ſame Ratio, as the Diameters of their Baſes; therefore Similar Cylinders and Cones, are in the Triplicate Ratio, or as the Cubes of their Altitudes.

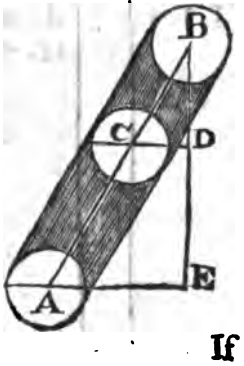
## PROP. XIII.

*If a Cylinder* A B, *be cut by a Plane* C, *parallel to its Baſe* A, *the Cylinders made by that Section, will be to one another, as the Parts of the Axis* A C, C B.

## PREPARATION.

Having let fall the Right-line B E, perpendicular to the Plane of the Baſe A, which will be alſo perpendicular to the Plane of the Parallel Baſe C, let the Lines A E, C D, be made the common Sections of the two Parallel Planes A, C, and the Triangular Plane B A E; ſo (16. 11.) will the two common Sections A E, C D, be parallel. Let A, C, the equal Circular Baſes of the Cylinders, be (*Cor.* 2. *Lem.* 2.) expreſſed by $\frac{1}{2}$XZX, and their Altitudes E D, D B, by A and *a*; ſo (*Cor.* 1. *Def.* 2.) will the Cylinders be among themſelves as $\frac{1}{2}$A X Z X to $\frac{1}{2}a$ X Z X.

I ſay, $\mathfrak{AC} : \mathfrak{CB} :: \frac{1}{2}\mathfrak{AXZX} : \frac{1}{2}\mathfrak{aXZX}$.




## DEMONSTRATION.

1. ED : DB :: AC : CB. 2. 6.
2. i. e. A : *a* :: AC : CB.
3. Alſo A : *a* :: $\frac{1}{2}$AXZX : $\frac{1}{2}a$XZX. *Lem.* 2. 5.
4. ∴ e. 2. 3. $\mathfrak{AC} : \mathfrak{CB} :: \frac{1}{2}\mathfrak{AXZX} : \frac{1}{2}\mathfrak{aXZX}$. 11. 5.

Q. E. D.

If the Cylinders made by the Section are Right, then their Axes are their Altitudes A, *a*, and conſequently (*Lem.* 2. 5.) $A : a :: \frac{1}{3} A X Z X : \frac{1}{3} a X Z X$.

## PROP. XIV.

*Cylinders and Cones, which are conſtituted upon equal Baſes, are to one another as their Altitudes.*

## PREPARATION.

Let the equal Circular Baſes of the Cylinders and Cones, be (*Cor.* 2. *Lem.* 2.) expreſſed by $\frac{1}{3} X Z X$, and their Altitudes by A and *a*; ſo (*Cor.* 1. *Def.* 2.) will the Cylinders be among themſelves, as $\frac{1}{3} A X Z X$ to $\frac{1}{3} a X Z X$, and (*Cor.* 10.) the Cones as $\frac{1}{9} A X Z X$ to $\frac{1}{9} a X Z X$.

I ſay, $\mathfrak{A} : \mathfrak{a} :: \frac{1}{3} \mathfrak{A} \mathfrak{X} \mathfrak{Z} \mathfrak{X} : \frac{1}{3} \mathfrak{a} \mathfrak{X} \mathfrak{Z} \mathfrak{X} :: \frac{1}{9} \mathfrak{A} \mathfrak{X} \mathfrak{Z} \mathfrak{X} : \frac{1}{9} \mathfrak{a} \mathfrak{X} \mathfrak{Z} \mathfrak{X}$.

## DEMONSTRATION.

1 $A : a :: \frac{1}{3} A X Z X : \frac{1}{3} a X Z X$. *Lem.* 2. 5.

Alſo 2 $A : a :: \frac{1}{9} A X Z X : \frac{1}{9} a X Z X$. *Lem.* 2. 5.

∵ e 1. 2. 3 $\mathfrak{A} : \mathfrak{a} :: \frac{1}{3} \mathfrak{A} \mathfrak{X} \mathfrak{Z} \mathfrak{X} : \frac{1}{3} \mathfrak{a} \mathfrak{X} \mathfrak{Z} \mathfrak{X} :: \frac{1}{9} \mathfrak{A} \mathfrak{X} \mathfrak{Z} \mathfrak{X} : \frac{1}{9} \mathfrak{a} \mathfrak{X} \mathfrak{Z} \mathfrak{X}$. 11. 5. Q. E. D.

PROP.

## PROP. XV.

*Equal Cylinders* BH, EK, *and Cones* BAC, EDI, *have their Baſes and Altitudes reciprocal; ſo that as* BC:EI:: DF:AG. *And thoſe Cylinders* BH, EK, *and Cones* BAC, EDI, *which have their Baſes and Altitudes reciprocal, are equal.*

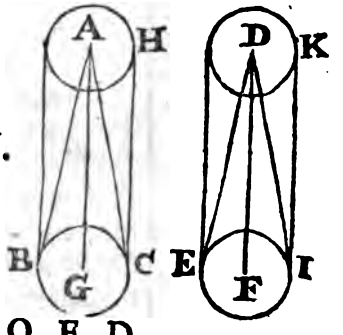
*Hyp.* 1 Suppoſe BH = EK. BAC = EDI.

*Caſe* 1 If the Cylinders BH, EK, and Cones BAC, EDI, be Right, and have their Altitudes DF, AG, equal.

### PREPARATION.

DF being (*Hyp.*) = AG, and BH = EK, the Ratio of DF to AG, is the ſame with that of BH to EK, and conſequently (*Def.* 2. 5.) DF:AG::BH:EK.

I ſay, BC:EI::DF:AG.




### DEMONSTRATION.

1 DF = AG. *Hyp.*
∵ e. 1. 2 BC:EI::BH:EK. 11.
Alſo 3 DF:AG::BH:EK. *Def.* 2. 5.
∵ e. 2. 3. 4 BC:EI::DF:AG. 11. 5. Q. E. D.

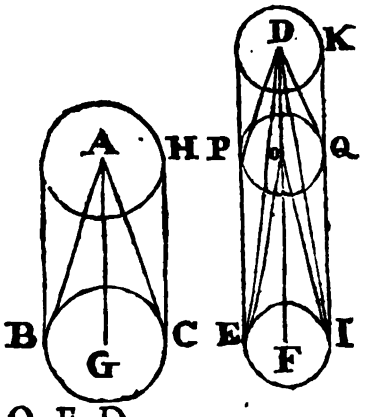
If

*Caſe.* 2 If the Cylinders BH, EK, and Cones BAC, EDI, be Right, and have their Altitudes DF, AG, unequal.

PREPARATION.

From the Greater DF, take OF = AG, and thro' O, draw PQ || EI.

I ſay, BC : CI :: DF : AG.




DEMONSTRATION.

| | | |
|---|---|---|
| | 1 | DF : OF :: EK : EQ. 14. |
| And | 2 | OF = AG. *Conſt.* |
| ∵ e. 1. 2. | 3 | DF : AG :: EK : EQ. *Sub.* |
| Again | 4 | BH = EK. *Hyp.* |
| ∵ e. 3. 4. | 5 | DF : AG :: BH : EQ. *Sub.* |
| Alſo | 6 | BC : EI :: BH : EQ. 11. |
| ∵ e. 6. 5. | 7 | BC : CI :: DF : AG. 11. 5. Q. E. D. |

*Caſe* 3 If the equal Cylinders and Cones be Inclined; their Baſes and Altitudes will be reciproal.

DEMONSTRATION.

Imagine Right Cylinders to be placed on the Baſes of the Inclined Cylinders, of the ſame Heights with them; ſo *(Cor. 2. Def. 2.)* will they be equal to the equal Inclined Cylinders, and conſequently to one another. Since therefore thoſe equal Right-Cylinders, have *(Caſe 1. and 2.)* their Baſes and Altitudes reciprocal, the Inclined Cylinders, their Equals, as alſo the Inclined Cones, (which *(10.)* are the Thirds of the equal Inclined Cylinders, of the ſame Baſes and Altitudes,) will have their Baſes and Altitudes reciprocal.

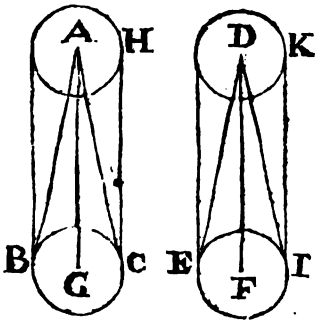
Sup-

*Hyp.* 2 Suppoſe BC:EI::DF:AG.

*Caſe* 1 If the Cylinders BH, EK, and Cones BAC, EDI, be Right, and have their Altitudes DF, AG, equal.

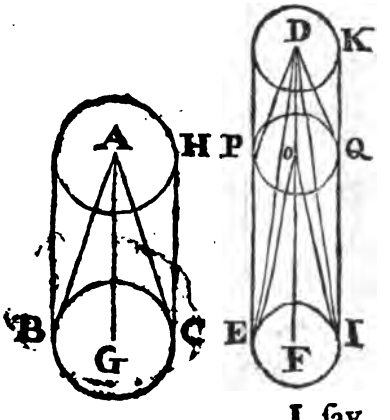
I ſay, 𝔅𝔜 = 𝔈𝔎. 𝔅𝔄𝔇 = 𝔈𝔇𝔍.

DEMONSTRATION.




| | | |
|---|---|---|
| | 1 | DF = AG. *Hyp.* |
| ∵ e. 1. | 2 | BC:EI::BH:EK. 11. |
| Alſo | 3 | BC:EI::DF:AG. *Hyp.* |
| ∵ e. 2. 3. | 4 | BH:EK::DF:AG. 11. 5. |
| ∵ e. 4. | 5 | BH:DF::EK:AG. 16. 5. |
| ∵ e. 5. 1. | 6 | BH:AG::EK:AG. *Sub.* |
| ∵ e. 6. | 7 | BH = EK. 9. 5. |
| 7 ÷ 3 | 8 | ⅓BH = ⅓EK. *Ax.* 14. |
| Alſo | 9 | ⅓BH = BAC, ⅓EK = EDI. 10. |
| ∵ e. 8. 9. | 10 | BAC = EDI. *Sub.* |
| ∵ e. 7. 10. | 11 | 𝔅𝔜 = 𝔈𝔎. 𝔅𝔄𝔇 = 𝔈𝔇𝔍. Q. E. D. |

*Caſe* 2 If the Cylinders BH, EK, and Cones BAC, EDI, be Right, and have their Altitudes DF, AG, unequal.

PREPARATION.

From the Greater DF, take OF = AG, and thro' O, draw PQ ‖ EI.